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# THE MATHEMATICS TEACHER

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## Using Algebra in Teaching Geometry\*

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### INTRODUCTION

IT HAS been frequently stated that we should use algebra to a greater extent in teaching geometry. The purposes for doing this have not always been made explicit. It may be a desire to prevent the loss of algebraic manipulations through lack of use (forgetting) and so having to reteach much of the elementary algebra in advanced courses. It may be a desire to have fused courses, that is to have algebra and geometry taught as one structure. If so, we must recognize that a new set of postulates must be established, for surely those of Euclidean geometry are distinct and based on different elements than those of abstract algebra. This may prove too difficult to master for any except the very brightest high school pupils. It may also be a desire to have algebra and geometry grow as separate structures, based on separate fundamental postulates, definitions, and theorems, but to show a sort of isomorphism between the structures (a correspondence of elements and operation). We shall take this as the soundest point of view.

Let us first of all dispose of many so-called algebraic proofs that are not algebraic, but merely use symbols common to arithmetic and algebra. Frequently these

proofs use results from algebra without appropriate reason. Consider the Pythagorean theorem, which is proved in many texts as follows:

1.  $c/b = b/x$  or  $cx = b^2$ ;  $c/a = a/y$  or  $cy = a^2$ . With reasons of proportionality of sides of similar triangles and theorems on properties of proportions.

2.  $a^2 + b^2 = cx + cy$  (Addition postulate). Then  $cx + cy$  becomes  $c(x + y)$  with no apparent reason given. What reason would you give, and is your reason a common geometric theorem or postulate? Of course it is easy to prove the sum of the rectangles

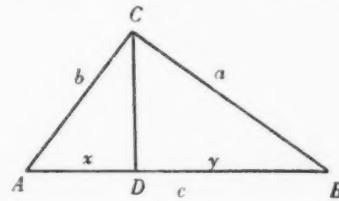


FIG. 1

$xc + cy$  is the same as the rectangle  $c(x + y)$ , and if  $x + y = c$ , the rectangle  $c(x + y)$  is the square on  $c$ ,  $c^2$ . The proof is then a geometric proof and could have been equally well established by using the usual notation  $AB$ ,  $AC$ ,  $BC$ , and  $AD$  and  $DB$ . By an algebraic proof however is meant, the designating of the size and position of geometrical configurations by numbers (or variables) operating on the numbers, and variables according to algebraic prin-

\* Presented at the Thirtieth Annual Meeting of the NCTM, April 18, 1952, at Des Moines, Iowa.

ciples, and finally reinterpreting the resulting numbers in terms of their geometrical counterpart.

#### GRAPHICAL REPRESENTATION

Perhaps the best and simplest approach to relating algebra and geometry in this manner is through the use of the function concept. To do this we make use of rectangular coordinates. Without actual measurement we can draw a graph of the variation of parts of a triangle as two parts are held constant. This was suggested and carried out by Beinhorn, a German textbook writer, in 1916. It was later reproduced in 1930 in the Fifth Yearbook of the National Council of Teachers of Mathematics by the late John Swenson<sup>1</sup>. We shall illustrate one case where a side and an adjoining angle are constant. The reader can try all the other various other possibilities.

In  $\triangle ABC$ , let side  $AB$  and angle  $A$  be constant, and let angle  $B$  increase continuously. We take regular values of angle

<sup>1</sup> John Swenson, "Graphic Methods of Teaching Congruency in Geometry," *The Teaching of Geometry*. (Fifth Yearbook, National Council of Teachers of Mathematics, [New York: Bureau of Publications, Teachers College, Columbia University, 1930]). 96-101.

$B$ , every  $15$  or  $20^\circ$  for convenience in graphing. We then graph side  $AC_n$  as a function of  $BC_n$ . In the initial study we need not take particular values but we note all the properties.

1.  $BC_n$  has no values less than the  $\perp$  from  $B$  to  $AC$ .
2.  $AC_n$  has two values for values of  $BC$  greater than  $\perp$ , but less than  $AB$ .
3. For values of  $BC_n$  equal to  $AB$  there is a zero and a positive value to  $AC$ .
4. For values of  $BC_n > AB$ , there is only one positive value to  $AC$  (and one negative also).

Thus the graphical study of the triangle gives us the entire story of the so-called ambiguous case in geometry of *ssa*. It also suggests a relationship to negative values for sides of a triangle. It also suggests a curve and its equation as the study of relations in the triangle. While all this study can come at a later date, for the reader it is easy to see that if  $\angle A = 45^\circ$ ,  $AB = 1$ , and we designate  $BC$  by  $x$ ,  $AC$  by  $y$ , we obtain as the equation of the curve  $x^2 - (y^2 - \sqrt{2}y) = 1$ , or more generally

$$x^2 - (y^2 - 2ay \cos A) = a^2.$$

We can now determine the possibility of one, two, or no real triangles by substitut-

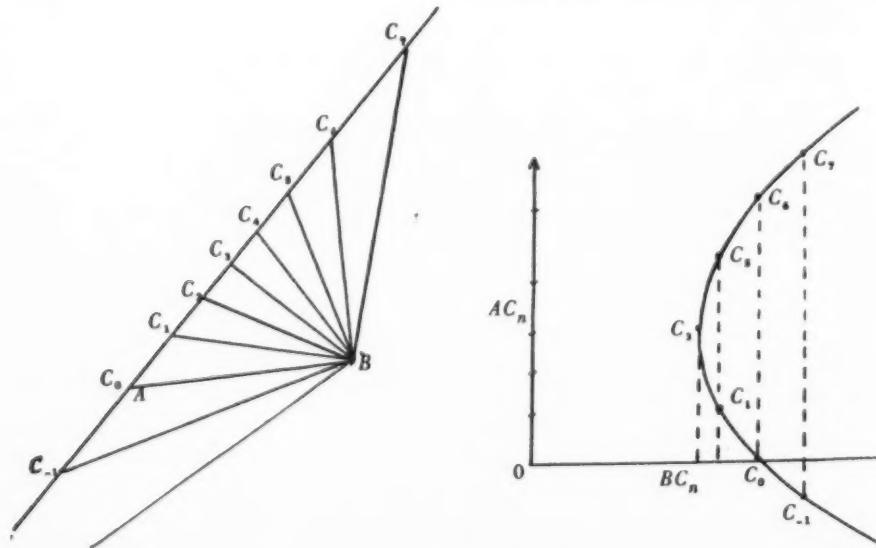


FIG. 2

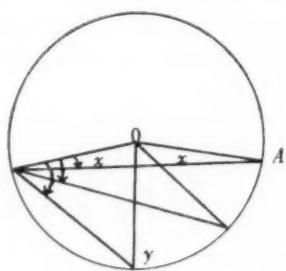


FIG. 2

ing numbers in the equation (for  $x, a, A$ ) and interpreting the resulting solution for  $y$ . This is studying geometry through the use of algebra.

Variations of the above figure are: to graph

- $AC_n$  as a function of  $\angle B_n$
- $BC_n$  as a function of  $\angle B_n$
- $AC_n$  or  $BC_n$  as a function of angle  $C_n$

In every case the graph can be studied to give the corresponding relations that hold in the triangle. The following examples are cited as further examples of the use of graphs in studying geometric relations.

1. The variation in parts of a triangle when two sides are held constant.
2. The variation in parts of a triangle when an angle and opposite side are held constant.
3. The variation in the base of a triangle as the altitude increases.
4. The variation in the length of a chord of a circle as a function of its distance from the center.
5. The variation in the arc of a circle as an inscribed angle (or any other type of angle) varies in size.
6. The variation in the area (perimeter) of figures of fixed perimeter (area).

In all cases it is an interesting and worth while project to determine, if possible, the algebraic formula for the function represented by the graph; to study the algebraic function, and thus discover other geometric relations. E.g., plot the arc of a central angle as a function of base angles at the chord of the arc.

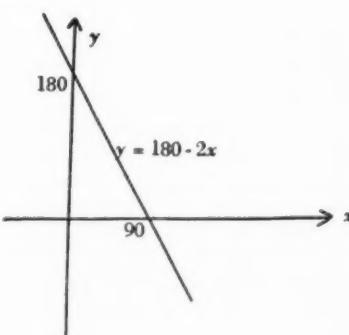


FIG. 3

It is easy to establish the relation  $y=180-2x$ . A study of this equation shows that if  $y=0$ ,  $\angle x=90^\circ$ . Hence a tangent is perpendicular to its radius.

#### ANALYTIC GEOMETRY

To define plane analytic geometry in rigorous fashion is quite a chore. We can, however, look upon it as a correspondence of points of a plane to ordered pairs of numbers; sets of points in a plane to relationships between the numbers of the ordered pairs. In a rectangular coordinate system we attach a certain meaning to the  $x$  and the  $y$ , in the ordered pair  $(x, y)$  that enables us to locate one and only one point in exactly one way. A set of points that have common properties are given names, and the set of points is usually determined by citing algebraic (or other) relationships between the  $x$  and  $y$  coordinates. We can thus define or establish by proof the concepts of line (or with the use of a parameter) line segment; any point on a line segment (internal or external); projection of a line segment; length of a line segment; slope of a line segment; perpendicular lines, etc. We can then use these properties to prove or discover other geometric relationships. We can also use this process to determine the possibility or impossibility of a Euclidean construction. An illustration or two from each of these aspects will suffice.

To prove the medians of a triangle meet at a point two-thirds distance of the median from the vertex.

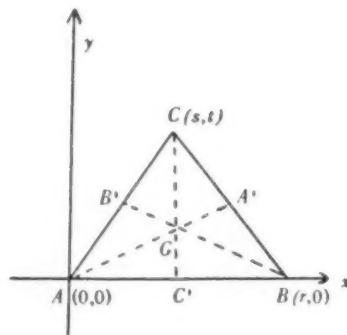


FIG. 4

In the triangle  $ABC$ , select  $A$  as the origin and  $AB$  as the axis of the independent variable. Draw the axis  $AY \perp AX$ . The points  $A$ ,  $B$  and  $C$  have fixed coordinates which are represented by  $(0, 0)$ ,  $(r, 0)$  and  $(s, t)$  respectively. The mid-point  $C'$  of  $AB$  has coordinates  $(\frac{1}{2}r, 0)$ . Similarly the midpoints  $A'$  and  $B'$  have the coordinates  $(\frac{1}{2}(r+s), \frac{1}{2}t)$  and  $(\frac{1}{2}s, \frac{1}{2}t)$  respectively. The points that divide  $AA'$ ,  $BB'$  and  $CC'$  into the ratio  $2:1$  are the coordinates  $(r+s/3, t/3)$ . Hence this is interpreted as the point of coincidence of all three medians.

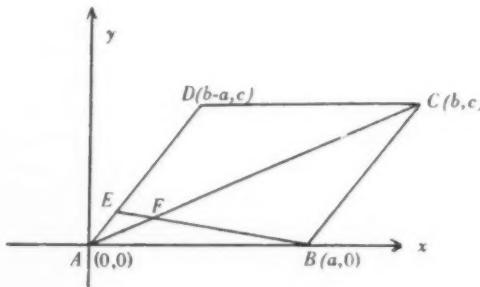


FIG. 5

Another example. In parallelogram  $ABCD$ ,  $AD$  is divided at  $E$  into the ratio  $1:n-1$  (that is  $AD$  is  $n \cdot AE$ ).  $BE$  cuts diagonal  $AC$  at  $F$ . What is the relation of  $AF$  to the whole diagonal? It is easy to show the coordinates of  $E$  are  $(b-a/n, c/n)$ , the equations of  $EB$  and  $AC$  are obtained and the coordinates of intersection are  $(b/n+1, c/n+1)$ . This is translated into the geometric relation:  $AF$  is contained into  $AC$  exactly  $n+1$  times. Of

course the same relationship can be obtained by ordinary geometry, but algebra supplies a unique method applicable to all problems, although not always the simplest solution.

In recent years, there has been much discussion in THE MATHEMATICS TEACHER on *Nedians*.<sup>2</sup> This whole discussion probably began as a result of a statement in

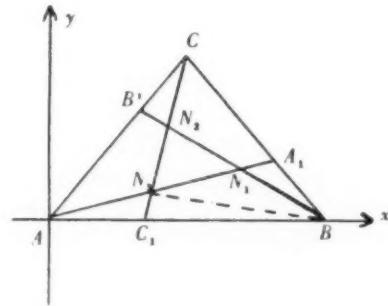


FIG. 6

H. Steinhaus' *Mathematical Snapshots* where he says it is well known that if lines are drawn from each vertex, trisecting the opposite side, the interior triangle is  $\frac{1}{7}$  of the original triangle. If you have not done so, try a strictly Euclidean proof. Then use coordinate geometry finding the coordinates of  $A'$ ,  $B'$ ,  $C'$ , the equations of  $AA'$ ,  $BB'$ , and  $CC'$ , and then the coordinates of  $N_1$ ,  $N_2$ , and  $N_3$ . Each median is at once seen to be divided into the ratio  $3:3:1$  and then the proof is easily noticed. We could also use the determinant formula for the area of a triangle to determine the same relation.

#### CONSTRUCTIBILITY

The Euclidean trisection of an angle, and other Euclidean constructions (by straight edge and ruler alone) are proved impossible or possible by the use of analytic geometry. These proofs are well known. It suffices to say here that any Euclidean construction must lead to a finite number of rational operations and extractions of real square roots. To illustrate, first consider the construction of a

<sup>2</sup> For the references see "Mathematical Miscellanea" page 603 of this issue.

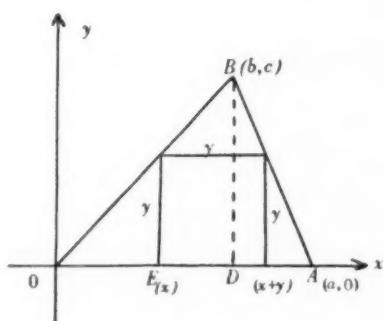


FIG. 7

square inscribed in a given triangle. In the figure, since the triangle is given we know points  $A(a, 0)$ ,  $B(b, c)$ ; that is we know the lengths  $OA$ ,  $OD$ ,  $DB$ . Let  $EF(y)$  be the side of the required square. From the figure (or past knowledge) we have  $y/a = c - y/c$ . Hence  $y = ac/a + c$ . Since this involves rational operations only on known elements,  $y$  is constructible, and hence the problem is possible in a Euclidean sense.

Now consider the problem to construct a triangle, given the distances of the orthocenter from the three vertices. To make the problem simple we select the origin at the orthocenter with one altitude on the  $x$ -

that of  $BE = -x/\sqrt{s^2 - x^2}$ . Slope of  $AC$  also equals  $-\sqrt{t^2 - x^2}/x - r$ . Hence  $x/\sqrt{s^2 - x^2} = \sqrt{t^2 - x^2}/x - r$  or  $2rx^3 - x^2(r^2 + s^2 + t^2) + s^2t^2 = 0$ . Substituting for  $r$ ,  $s$ ,  $t$  the given values we find  $4x^3 + 29x^2 - 144 = 0$  which is irreducible because this equation has no rational root and hence  $x$  is not constructible. The Euclidean solution is therefore impossible. Incidentally, one set of approximate answers is  $x = 1.9754$  making  $m = 1.3169$  and  $n = .9877$ . This simple algebraic analysis thus prevents wasting hours and hours of time seeking an impossible Euclidean construction. On the other hand given an altitude from one vertex ( $AD = k$ ) and the distance of the other two vertices from the orthocenter ( $s$  and  $t$ ) results in a reducible quartic equation in  $x$  and hence  $x$  can be found, and the construction made.

#### STRAIGHTFORWARD ALGEBRA

The use of algebraic processes is quite common to high school teachers of geometry. Consider, for example, the development of Hero's formula for the area of a triangle. In this derivation, each necessary line segment is designated by a number

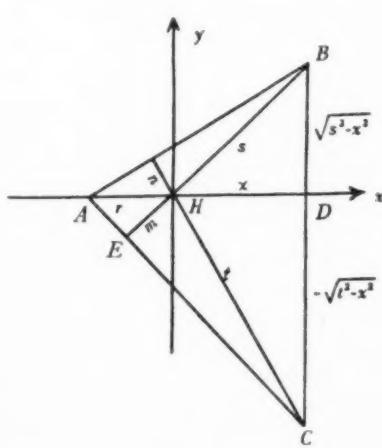


FIG. 8

axis. We know the three values  $AH = r = -2$ ,  $BH = s = 3$ ,  $CH = t = 4$ . In the figure let  $HD = x$ . Then  $BD = \sqrt{s^2 - x^2}$ ,  $DC = -\sqrt{t^2 - x^2}$ . Slope of  $BE = \sqrt{s^2 - x^2}/x$  and the slope of  $AC =$  negative reciprocal of

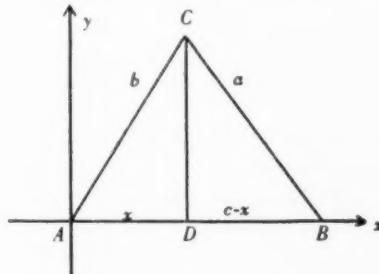


FIG. 9

which represents its length. (Note that by establishing coordinates, it is an easy matter to establish these lengths.) The numbers are then subjected to the usual operations (theorems) of algebra. Thus by the definition of length,  $b^2 = h^2 + x^2$ ;  $a^2 = h^2 + (c-x)^2$ ; by algebra  $(c-x)^2 = (c-x)c + (c-x)x = \text{etc.} = c^2 - 2cx + x^2$ ; by substitution,  $a^2 = b^2 + c^2 - 2cx$ , and by the theorems of algebra  $x = b^2 + c^2 - a^2/2c$ . It is an excellent exercise to pause here awhile

and explain the geometrical interpretation of this expression for  $x$ . Substituting in  $h^2 = b^2 - x^2$  we apply rule after rule (theorem after theorem) obtaining

$$\begin{aligned} 4h^2c^2 &= 4c^2b^2 - (b^2 + c^2 - a^2)^2 \\ &= (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2) \\ &= [(b+c)^2 - a^2][a^2 - (b-c)^2] \end{aligned}$$

= and so on as the teacher of plane geometry knows.

The important fact here is that algebraic theorems are applied and the final result interpreted in terms of the original geometric elements they represent.

We use a similar process in finding the volume of a frustum of a pyramid. Let the

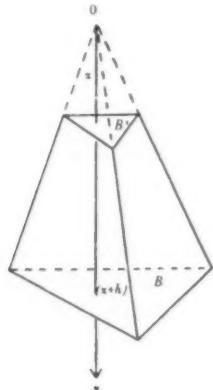


FIG. 10

altitude be on the  $OX$  axis with vertex at  $O$  meeting base  $B'$  at a distance  $x$  and base  $B$  at a distance  $x+h$ . Then

$$\frac{x}{x+h} = \frac{\sqrt{B'}}{\sqrt{B}} \text{ and } x = \frac{h\sqrt{B'}}{\sqrt{B} - \sqrt{B'}}.$$

For the frustum  $V = \frac{1}{3}B(x+h) - \frac{1}{3}B'x$ . Substituting for  $x$  and performing the necessary algebraic operations we find  $V = \frac{1}{3}h(B + \sqrt{BB'} + B')$ . This is now interpreted geometrically as the volume of three pyramids, all having the same altitude, but with bases equal to the lower base, the upper base, and the mean proportional between these bases.

One final example of straightforward algebra should give a cue to a host of pos-

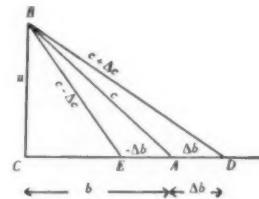


FIG. 11

sibilities. In right triangle  $ABC$ , let side  $CA = b$  be increased by an amount  $\Delta b$ . Then  $CD = b + \Delta b$ . Also  $BD$  is longer than  $c$  say  $c + \Delta c$ . Now in  $\triangle BCD$ ,  $c^2 + (b + \Delta b)^2 = (c + \Delta c)^2$  or  $(c + \Delta c)^2 = c^2 + \Delta b^2 + 2b \Delta b$ . This last statement interpreted geometrically for  $\triangle BAD$  gives the usual theorem on the square of a side opposite an obtuse angle. Similarly, using  $\triangle BCE$ ,  $(c - \Delta c)^2 = a^2 + (b - \Delta b)^2$  or  $(c - \Delta c)^2 = c^2 + \Delta b^2 - 2b \Delta b$  and the geometric interpretation is evident.

Thus straightforward algebra can aid us in deriving theorems in geometry.

It seems necessary to add a note of caution in teaching both algebra and geometry indiscriminantly in the same course. Unless done with the utmost care and the clearest formulation of purpose, it can lead to confusion, misunderstanding, and complete lack of organization. Let us be clear about the following: (1) Geometry can be developed synthetically, on the foundation of a set of axioms (such as Hilbert's) without any recourse whatsoever to algebra. This is the usual treatment in high school geometry and it has a simple and an important service to render in developing the concept of what mathematics is. (2) Algebra (of the real numbers or of complex numbers) can be developed deductively, on the foundation of a set of axioms (such as Peano's or Russell's or Huntington's) and while more abstract in nature since it is harder to find a physical model to explain all its theorems, it too renders an important service. Each of these two areas, synthetic geometry and algebra of the real number, are distinct mathematical discipline and should be

(Continued on page 571)

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† See "Geometry," by

issue.

# Using Geometry in Teaching Algebra\*

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ALGEBRAIC CONCEPTS may be used in defining elements of geometry, and geometric concepts may be used in defining elements of algebra. These alternatives, along with discussions of the use of algebra in teaching geometry<sup>†</sup> and the use of geometry in teaching algebra, should not be considered as creating a problem similar to the proverbial problem of the chicken and the egg. Rather they indicate how the interdependence of algebra and geometry may be used to improve the presentation and understanding of both subjects in terms of fundamental concepts of mathematics.

Euclidean geometry may be developed from points, lines and relations among them. An algebra of real numbers may be developed from the positive integers and relations among them. In each of these developments there must be undefined elements such as points, lines, and positive integers; there must be postulates or assumed relations among these elements; and there must be definitions of new elements in terms of those assumed. In each development one must postulate that the elements are ordered [ $a$  precedes  $b$ ], the elements are dense [between any two elements there is a third element]; and the elements are continuous. Continuity may be postulated in both geometry and algebra in terms of limits of sequences of elements. In geometry continuity may also be postulated by assuming that every line segment joining the center of a circle to a point outside the circle contains a point of the circle. This is equivalent to assuming that every curve joining points on opposite

sides of a line intersects the line. In algebra continuity may also be postulated by assuming that every infinite decimal represents a number.

These concepts of undefined elements, postulates, definitions, order relations, existence of an element between any two elements, continuity—lie at the foundations of mathematics. Along with other basic concepts they may be used very effectively to improve the high school student's understanding of mathematics. They provide the basis for presenting fundamental operations and concepts of both algebra and geometry simultaneously along with their applications each upon the other and in the practical uses of mathematics in our culture.

With this introduction let us now consider the use of geometry in teaching algebra. We shall be primarily concerned with an algebra of points upon a line.

All algebraic processes are based upon sets of elements usually called numbers. Positive integers and positive rational numbers are easily understood without using geometric concepts. Negative integers, negative rational numbers, and irrational numbers are easily visualized in terms of geometric concepts. We shall use geometric constructions for addition, subtraction, multiplication, division and the extraction of square roots to emphasize the underlying properties of rational and irrational numbers. This geometric point of view can be very helpful in presenting these concepts in secondary school.



FIG. 1

We first select an origin and a unit point on the line (Fig. 1). By convention the unit point is taken to the right of the

\* Presented at the Thirtieth Annual Meeting of the NCTM, April 18, 1952, at Des Moines, Iowa.

† See also "Using Algebra in Teaching Geometry," by Howard F. Fehr, pp. 561-66 of this issue.

origin. This is not necessary, but it does, however, emphasize the role of the origin and the unit point in designating the positive sense or direction along the line as well as the unit of magnitude.

We now associate positive integers with points on the line obtained by marking off units to the right of the origin (Fig. 2).



FIG. 2

Throughout this paper we shall consider the points and the symbols (numbers) associated with them interchangeably. Technically the two sets of elements are isomorphic, i.e., the one-to-one correspondence between the set of points and the set of numbers is preserved under addition and multiplication.



FIG. 3

Given any two numbers  $a, b$  we define  $a < b$  if  $a$  is to the left of  $b$  (Fig. 3),  $a = b$  if  $a$  and  $b$  coincide,  $a > b$  if  $a$  is to the right of  $b$ .

We associate  $a+b$  with a point  $b$  units to the right of  $a$  and consider  $ab$  as the sum of  $b$  elements  $a$ . The number  $ab$  may also be determined using similar triangles. We shall use similar right triangles (Fig. 4).

Addition and subtraction are inverse operations in the sense that  $(a+b) - b = a$  for any positive integers  $a, b$ . Since  $a+b$  is  $b$  units to the right of  $a$ ;  $(a+b) - b$  is  $b$  units to the left of  $a+b$ . Accordingly we define  $0 - b$  as  $-b$  and associate negative integers with units marked off to the left of the origin.

Addition and subtraction of positive and negative integers may be easily visualized in terms of marking off units in appropriate directions on the line. All algebraic properties of addition and subtraction of signed integers may be interpreted on the number line.

Multiplication of signed integers may be

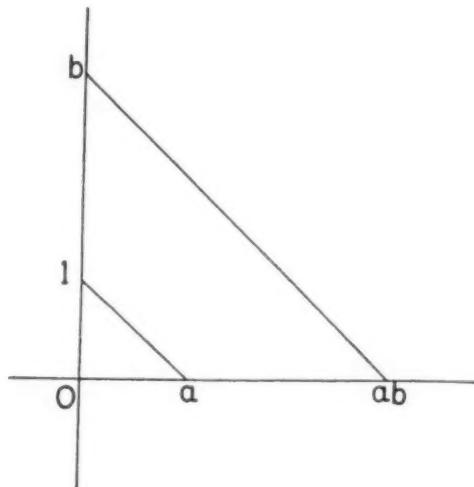


FIG. 4

interpreted either as repeated addition or in terms of similar triangles. When similar triangles are used, the order relations in geometry imply the usual order relations for the numbers. For example, if  $a$  and  $b$  are any two positive integers, we may take two axes intersecting at right angles, take the point of intersection as  $0$  on both axes, determine  $a$  on one axis, determine  $1$  and  $b$  on the other, draw the line joining  $1$  and  $a$ , and determine  $ab$  by drawing a parallel line through  $b$  as in Figure 5. We

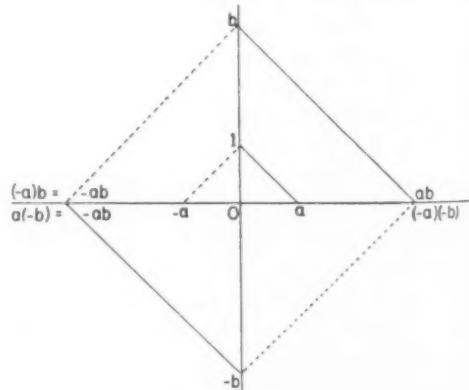


FIG. 5

next determine  $-b$  and  $a(-b)$  as in Figure 5 and show by congruent triangles that under the postulates of our geometry  $a(-b) = -ab$ . Similarly using  $-a$  we may show

that  $(-a)b < 0$ ,  $(-a)b = -ab = a(-b)$ ,  $(-a)(-b) > 0$  and  $(-a)(-b) = ab$ . Thus the usual conventions regarding signs of products may be established geometrically. The properties of multiplication by zero and unity may be similarly established. These geometric interpretations cannot, however, be considered as algebraic proofs. They merely indicate that the above conventions and properties are consequences of our assumed postulates for geometry.

Division may be defined as the inverse of multiplication in the sense that  $(ab) \div b = a$  when  $b \neq 0$ . Under the assumption that  $b \neq 0$  we may construct  $a/b$  for any integer  $a$  as follows: determine 0 and 1 on the number line axis, draw a second axis intersecting the number line at 0, on the second axis take 0 coinciding with the 0 on the first axis and determine 1,  $a$  and  $b$ ; draw the line from  $b$  to 1 as in Figure 6,

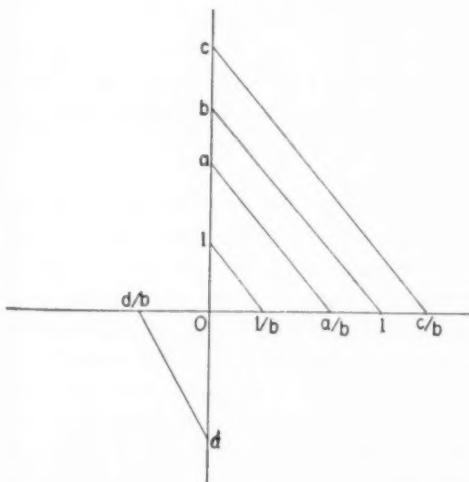


FIG. 6

draw a parallel line through 1 to obtain  $1/b$  and in general through  $a$  to obtain  $a/b$ . When  $b > 0$ ,  $1/b > 0$ . If  $0 < a < b$  then  $0 < a/b < 1$ . If  $0 < b < c$  then  $1 < c/b$ . If  $d < 0$  and  $0 < b$  then  $d/b < 0$ . If we were to construct  $1/0$ , we would first join 0 to 1 (this line coincides with the number line) and then draw a line through the unit point on the second axis parallel to the line

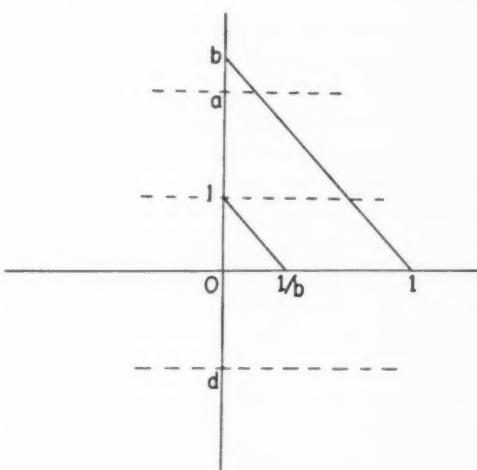


FIG. 7

just drawn (Fig. 7). Since the line through the unit point is parallel to the number line, these lines do not intersect and  $1/0$  is undefined. Similarly  $a/0$  and  $d/0$  are undefined whenever  $a > 0$  or  $d < 0$ . The symbol  $0/0$  is said to be *indeterminate* since the line through 0 parallel to the first line coincides with the number line and therefore does not determine a unique point of intersection. The distinction between undefined and indeterminate symbols may thus be visualized geometrically as the distinction between no point of intersection and no uniquely determined point among many points of intersection.

We may now construct all rational numbers, i.e., all quotients  $a/b$  of integers where  $b \neq 0$ . This development of the rational number system is necessarily equivalent to that studied in algebra. It may be used to strengthen the students' algebraic concepts. All properties of rational numbers may be illustrated geometrically at the secondary level. They may be formally proved at the college level.

We start with an *origin* and a *unit point* on a *line*. We obtained *positive integers* by adding units. We obtained *negative integers* by subtracting units, and we obtained *rational numbers* by multiplying and dividing integers.

The rational numbers are *dense*, i.e., between any two distinct rational num-

bers there is a third rational number. The line appears to be covered with rational points and yet there are points on the line such as  $\sqrt{2}$  that are not rational. The  $\sqrt{2}$  may be constructed as the diagonal of a unit square; it may be represented as the limit of an infinite sequence of rational numbers 1, 1.4, 1.41, 1.414, . . . , or it may be represented as an infinite decimal 1.414214 . . . . Since all non-zero rational multiples of  $\sqrt{2}$  are also irrational, there are infinitely many "holes" in the line of rational points—holes in the sense that a curve may cross the line without intersecting it in a point corresponding to a rational number.

Suppose we consider all points on the line constructible from the unit distance using ruler and compasses, i.e., all points expressible in terms of integers using a finite number of the operations  $+$ ,  $-$ ,  $\cdot$ ,  $\div$  and extraction of square roots. Are there still "holes" in the number line?

Yes. The number  $\sqrt[3]{2}$  is not constructible. It is the intercept of the graph of  $y = x^3 - 2$ . Suppose we add all algebraic points—all points that are associated with roots of polynomials with rational coefficients. These numbers include all rational numbers, radicals, and since the roots of some equations of degree greater than four cannot be expressed in terms of radicals, they include other numbers also. Are there still "holes" in the number line?

Yes. If we put a spot of paint on a circle of unit diameter, place the circle with the painted spot on the origin and roll it without slipping along the line, the next paint spot on the line will be at  $\pi$  which is not an algebraic number. It can be represented as a limit of an infinite sequence of rational numbers 3, 3.1, 3.14, 3.141, . . . ,

or as an infinite decimal, 3.141593 . . . . Suppose we add all numbers expressible in terms of decimals (finite or infinite), i.e., all real numbers. Are there still "holes" in the number line?

No. We have now made our line continuous by including all limits of convergent sequences of rational numbers. There is a 1-1 correspondence between the real numbers and the points on the line in Euclidean geometry. Order relations, addition, multiplication, . . . still have their usual properties. Any curve crossing the line intersects the line in a point corresponding to a real number.

One might ask, can the real number system be extended in the same manner as we have extended the other sets of numbers? The answer is clear from the fact that there are no more "holes" in the number line and that our order relations depend upon the sense or direction in which one travels on the line. The real number system cannot be extended without changing the order relations. If the order relations are no longer considered, one may extend the real numbers to the complex numbers and many more abstract number systems.

Let us conclude our discussion of the use of geometry in teaching algebra with brief comments upon two other applications of the concept of continuity. Any polynomial in a real variable  $x$  with real coefficients has the property that if it is negative when  $x=a$  and positive when  $x=b$ , then its graph must cross the  $x$ -axis at least once between  $a$  and  $b$ . It may cross any odd number of times (Fig. 8). This property is used in the approximation of roots of polynomial equations.

Continuity is also very important in

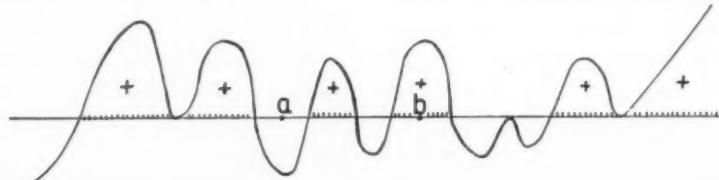


FIG. 8

any consideration of polynomial inequalities, such as  $p(x) > 0$ . A crude sketch of the graph of the polynomial in terms of its zeros and their multiplicities (Fig. 8) enables the student to recognize immediately the content of the inequality, i.e., the intervals on which the polynomial is positive.

We have considered a few examples of the use of geometric concepts in the teach-

ing of algebra. We have not taken time to discuss graphical solutions of problems, geometric interpretations of systems of simultaneous equations and many other important applications of the use of geometry in teaching algebra. The basic thesis of this paper has been that all branches of mathematics can be best understood in terms of their interdependence upon the foundations of mathematics.

### Using Algebra in Teaching Geometry

(Continued from page 566)

clearly kept so in the minds of our students. (3) Analytic geometry, a study of space through the use of algebra, also has a definite set of axioms. Here there is an isomorphism between elements (or combination of elements) in space and numbers (or groups and operations on numbers) assigned or given the name of the element. Thus  $(x, y)$  or  $(x, y, z)$  is a point;  $ax+by+c=0$  is a straight line;  $ax+by+cz+d=0$  is a plane;  $m_1 = -1/m_2$ , is a

condition of perpendicularity of lines;  $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$  is the length of a line segment; etc. The symbols are numbers, but they designate geometrical entities. And furthermore, under these analytic geometry agreements, the axioms of Euclidean synthetic geometry can be translated into equivalent algebraic statements, and hence all Euclidean geometry theorems (relations) are solvable by coordinate geometry. The process of doing this is given in the appendix of *Coordinate Geometry* by L. P. Eisenhart, a book all teachers should examine.

### Editor's Note

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## The Decimal Point and Slide Rule Answers

By E. L. EAGLE

Glenn L. Martin Company, Baltimore, Maryland

THE PLACING of the decimal point in the slide rule answer is frequently done through a process of estimating the approximate answer and making placement accordingly.<sup>1</sup> For some people, the method is sometimes time-consuming as well as productive of incorrect conclusions. Shuster<sup>2</sup> found that slide rule learners experience considerable difficulty in using this method—indeed, 33.4% of the errors made by the students in his study came from this source. Shuster recommended the “standard number” system<sup>3</sup> as being a more desirable method of meeting the problem of the straying decimal point. A method, arising out of the logarithmic nature of the slide rule, is presented here. This method is explained only in connection with its application to the C- and D-scales. It may, however, be extended to include all scales on the Log Log Duplex Trig slide rule.<sup>4</sup>

When common logarithms are used to make a computation, it is taken for granted that the use of the characteristic of the logarithm in placing the decimal point in the answer is a sound practice. No one apparently advocates that the use of the characteristic be discontinued, and that the decimal point should be placed in accordance with an approximate answer. Therefore, a system nearly as simple for the slide rule, a method which

<sup>1</sup> J. M. Thomas, “Pointing Off in Slide Rule Work,” *American Mathematical Monthly*, LV (November, 1948), 567.

<sup>2</sup> Carl N. Shuster, *A Study of the Problems in Teaching the Slide Rule*. (New York: Bureau of Publications, Teachers College, Columbia University, 1940.)

<sup>3</sup> E. W. Banhagel, *Logarithms and Slide Rule For Practical Use*. (Chicago: Ziff-Davis Publishing Company, 1945), p. 113.

<sup>4</sup> E. L. Eagle, The Log Log Scales of the Slide Rule. *Mathematics Magazine*, XXV (Nov.-Dec. 1951), 101.

makes use of the characteristic of the logarithms of the numbers involved in a computation, would likely be welcomed by many slide rule users. It is hoped this paper will do just that. The adoption of a method for pointing off in no way implies that one should not also check the reasonableness of each answer. In fact, one can (and should) develop his ability at sensing number size, an invaluable aid to confirmation of correctness in procedure and accuracy in computation.

Under the system presented here, one simply writes above (or below) each number used in a computation the characteristic of the logarithm of the number, unity being added to this characteristic when the slide extends to the left; and, at the end of the computation, these characteristics are added or subtracted (in accordance with the laws of logarithms for such operations) to obtain the characteristic of the answer.

A person who has used the slide rule for years will not find it easy to jot down the characteristic for each operation as it is completed. He is accustomed to carrying through all operations of a computation without devoting this moment between operations for designating characteristics. On the other hand, beginners do not experience this as a difficulty. Indeed, for them it becomes a device to show the operations completed at any time during a computation and to indicate the operations yet to be done.

This system requires the use of the definition of the characteristic of a logarithm and a few rules analogous to the laws of logarithms. While this article assumes the reader to be familiar with logarithms, it is noteworthy that such knowledge, though helpful, is not at all mandatory to the use of this method—that is, the use of the

definition of a characteristic and the rules which follow in this paper suffice to make this possible.

**Rule 1. Multiplication:** *The characteristic of the product of two numbers is the sum of the characteristics of those numbers (the characteristic of the second factor being increased by unity if the slide extends to the left).*

**Example 1.**

Evaluate  $(4.75)(0.00665)$ .

**Procedure:** Write the characteristic<sup>5</sup> of 4.75, namely 0, above it. Set the right C-index to 475 on the D-scale. Move the hairline to 665 on the C-scale. Since the slide extends to the left, write the characteristic of 0.00665 increased by unity, namely  $(-3+1)$ , above it. Take the reading: 316. Compute the characteristic of the answer:  $0+(-2)=-2$ . Thus,

$$(4.75)(0.00665) = .0316.$$

**Example 2.**

Evaluate  $(.00213)(115000)$ .

**Outline:**

$$(.00213)(115000) = 245.$$

**Rule 2. Division:** *The characteristic of the quotient of one number divided by another is found by subtracting the characteristic of the divisor from the characteristic of the dividend (the characteristic of the divisor being increased by unity when the slide extends to the left).*

**Example 3.**

$$\text{Evaluate } \frac{0.675}{0.0825}.$$

**Procedure:** Write the characteristic of 0.675, namely  $-1$ , above it. Set the hairline to 675. Draw 825 on the C-scale under the hairline. Since the slide extends to the left, write the characteristic of 0.0825 plus unity, namely  $(-2+1)$ , below it. Take the reading: 818. Compute the characteristic of the answer:  $-1-(-1)=0$ . Thus,

<sup>5</sup> Meaning: the characteristic of the logarithm of 4.75.

$$\frac{0.675}{0.0825} = 8.18.$$

**Rule 3. Compound Operations of Multiplication and Division:** *The characteristic of the answer is equal to the sum of the characteristics of the factors in the numerator less the sum of the characteristics of the factors in the denominator (the characteristic of each factor being increased by unity when the associated position of the slide is to the left).*

**Example 4.**

$$\text{Evaluate } \frac{(0.00786)(213)(18300)}{12.15}.$$

**Procedure:** Write the characteristic of 0.00786, namely  $-3$ , above it. Set the right C-index to 786 on the D-scale. Move the hairline to 213 on the C-scale. Since the slide extends to the left, write the characteristic of 213 plus unity, namely  $(2+1)$ , above it. Draw 1215 on the C-scale under the hairline. Write the characteristic of 12.15, namely 1, below it. Move the hairline to 183 on the C-scale. Write the characteristic of 18300, namely 4, above it. Take the reading: 252. Compute the characteristic of the answer:  $(-3+3+4)-1=3$ . Thus,

$$\frac{(0.00786)(213)(18300)}{12.15} = 2520.$$

#### DERIVATION OF THE SYSTEM

**The C-Scale.** The distance from the left index of the C-scale to any specific point on the C-scale with which is associated a particular number, say X (as designated by numbers printed in the vicinity of that point), represents the mantissa of X. This distance is always less than unity, and the distance from the left index to the right index is exactly 1 unit. So, the distance from the right index to X represents  $(1 - \text{mantissa of } X)$ . The same statements may be made about the D-scale, since it is

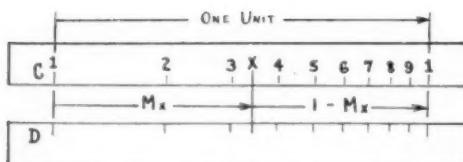


FIG. 1. The C- and D-scales.

identical to the C-scale. See Figure 1, in which  $M_x$  stands for "mantissa of  $X$ ."

*Multiplication.* Suppose we wish to multiply  $A$  by  $B$  to obtain the product,  $P$ :  $P = AB$ . Then, by logarithms,

$$\log P = \log A + \log B.$$

Adopting the nomenclature  $C_x$  to stand for the characteristic of  $X$ , and  $M_x$  for the mantissa of  $X$ , the above equation becomes

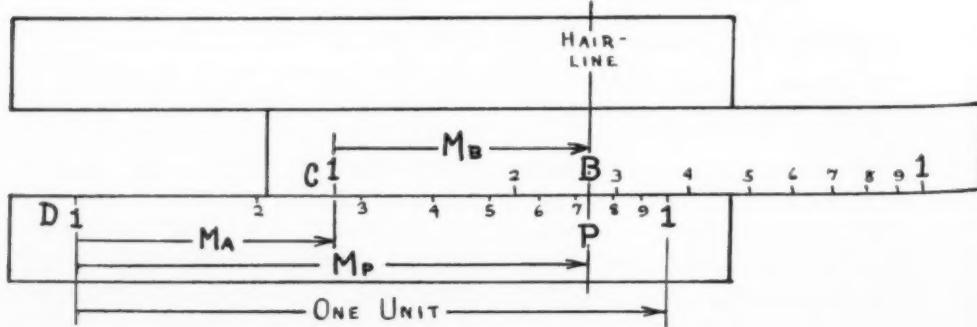
$$C_P + M_P = (C_A + M_A) + (C_B + M_B). \quad (1)$$

*Case I.* Experience in dealing with logarithms reminds one that in adding  $M_A$  and  $M_B$ , sometimes there is no carryover, this being the case when  $M_A + M_B < 1$ . If this is the case, then the distance for  $M_A$  taken on the D-scale plus the distance for  $M_B$  on the C-scale must give a total distance less than unity, and consequently the hairline must fall short of reaching the right D-index with the result that the slide extends to the right. There being no carryover,

$$M_P = M_A + M_B. \quad (2)$$

This conclusion is also verified by the addition of line segments in Figure 2. Subtracting (2) from (1), we have

$$C_P = C_A + C_B.$$

FIG. 2. Slide extends to right;  $M_P = M_A + M_B$ .

Thus, when the slide extends to the right, the characteristic of the product,  $P$ , equals the sum of the characteristics of the factors  $A$  and  $B$ .

*Case II.* If upon adding  $M_A$  and  $M_B$ , there is a carryover, then

$$1 < M_A + M_B < 2.$$

If we attempt to add, graphically, the distances for  $M_A$  and  $M_B$  as was done in Case I, the hairline will move beyond the D-index (and no reading on the D-scale can be made). So, to rectify this situation, the right C-index is moved to the former position of the left C-index, and the slide now extends to the left. Recalling that there was a carryover, we may state

$$M_P + 1 = M_A + M_B. \quad (4)$$

Or,

$$M_P = M_A + (M_B - 1). \quad (3)$$

The validity of relationship (3) is also supported by Figure 3. Subtracting (3) from (1),

$$C_P = C_A + (C_B + 1).$$

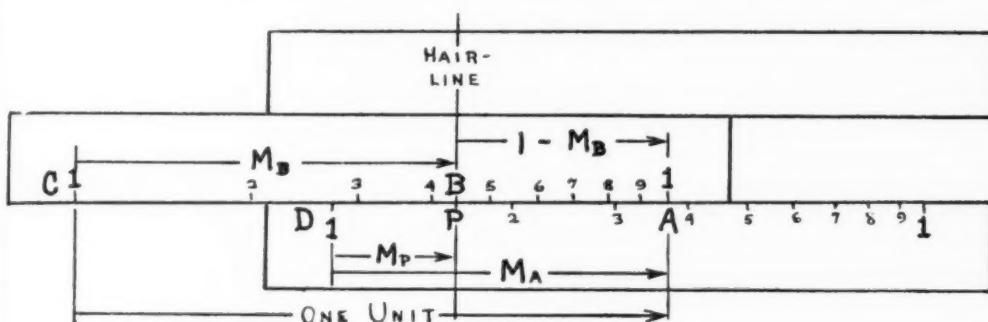
Thus, when the slide extends to the left, the characteristic of  $P$  is equal to the characteristic of  $A$  plus the characteristic of  $B$  increased by unity.

*Division.* Let  $A$  be divided by  $B$  to yield the quotient  $Q$ :  $Q = A/B$ . By logarithms,

$$\log Q = \log A - \log B.$$

Or, using symbols introduced earlier:

$$C_Q + M_Q = (C_A + M_A) - (C_B + M_B). \quad (4)$$

FIG. 3. Slide extends to left;  $M_P = M_A - (1 - M_B)$ 

Two cases are apparent: either  $M_A > M_B$  or  $M_A < M_B$ .

Case I. When  $M_A > M_B$ , it is unnecessary to borrow in order to subtract  $M_B$  from  $M_A$  as directed in the right member of (4). Hence,

$$M_Q = M_A - M_B. \quad (5)$$

Or,

$$M_A = M_Q + M_B.$$

Figure 4 shows a setting made on a slide rule for dividing  $A$  by  $B$ , and obviously

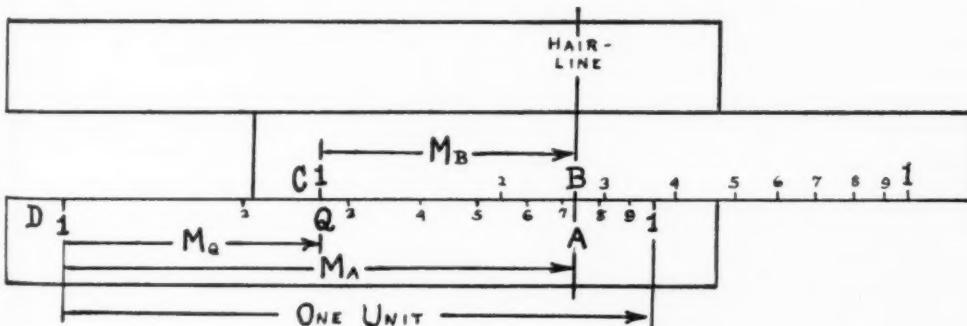
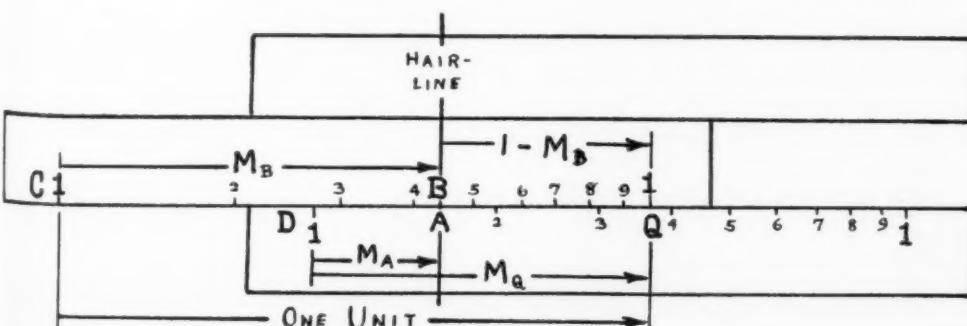
$M_A$  can be greater than  $M_B$  only when the slide extends to the right. Subtracting (5) from (4),

$$C_Q = C_A - C_B. \quad (6)$$

Thus, when the slide extends to the right, the characteristic of the quotient is equal to the characteristic of the numerator minus the characteristic of the denominator.

Case II. When  $M_A < M_B$ , it is necessary to borrow in order to subtract  $M_B$  from

(Continued on page 577)

FIG. 4. Slide extends to right;  $M_A = M_Q + M_B$ .FIG. 5. Slide extends to left;  $M_A = M_Q - (1 - M_B)$ .

# Leonardo da Vinci and the Center of Gravity of a Tetrahedron

By JOHN SATTERLY

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WE ARE all celebrating this year (1952) the quincentenary of the birth of Leonardo da Vinci (1452-1519), the great Italian artist, physicist, engineer and anatomist. One problem which he was the first to solve may be of interest to our younger mathematicians.

Archimedes (287-212 B.C.), the great Greek mathematician of antiquity, had been successful in determining the position of the center of gravity of plane figures and Leonardo advanced to solid figures solving the problem of the position of the center of gravity of a uniform tetrahedron or triangular pyramid. He stated the result in these words (I give a translation) "The center of gravity of a pyramid is in the fourth of its axis towards the base" and he accompanies this note with a diagram. His proof is, however, not given in full and maybe he arrived at the correct result by intuition. I shall supply here what may have been in his mind.

Archimedes had shown that the center of gravity of a plane triangular sheet was at the common intersection of the medians, this point (now also called the centroid) being one third of the way up a median from the corresponding base. Leonardo knew this and may have proceeded as shown in Figure 1.

Let  $PQRD$  (Fig. 1 similar to Leonardo's diagram) represent, in perspective, a tetrahedron and  $MD, MP$  the medians of the two faces  $DQR, PQR$  drawn from the common edge  $QR$ . The centers of gravity of the two faces  $DQR, PQR$  are at  $X$  and  $Y$  each one-third of the way up the medians  $MD, MP$  respectively. Join  $PX, DY$ . Triangular slabs of the tetrahedron cut parallel to the face  $DQR$  will naturally have their centers of gravity on  $PX$  and therefore the centers of gravity of the whole tetrahedron will be on  $PX$ . Simi-

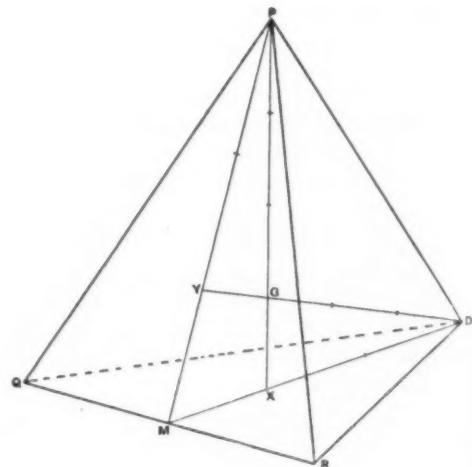


FIG. 1

larly it will lie on  $DY$ . Therefore it must be located at  $G$  the point of intersection of  $PX$  and  $DY$ . Leonardo states that  $G$  is one quarter of the way up  $XP$  towards  $P$  and one quarter of the way up  $YD$  towards  $D$ . He does not give a proof and he might have considered it an obvious result or he may have arrived at it by intuition or careful draftmanship.

Lesser folk, like ourselves, require a proof and I give here a proof which Leonardo may have followed: The section  $PMD$  of Figure 1 is reproduced in Figure 2. Join  $XY$ . Since  $MX$  and  $MY$  are each one-third of  $MD$  and  $MP$  respectively,  $XY$  is parallel to  $PD$  and in length equal to one-third of  $PD$ . Therefore, the triangles  $XGY$  and  $PGD$  are similar and  $XG/PG = XY/PD = YG/GD = 1/3$  whence it follows that  $G$  is one-fourth of  $XP$  up from  $X$  towards  $P$ , and also one-fourth of  $YD$  up from  $Y$  towards  $D$ . A similar result holds for the other faces and medians that might have been taken.

The result thus obtained by Leonardo for the triangular pyramid is true for all

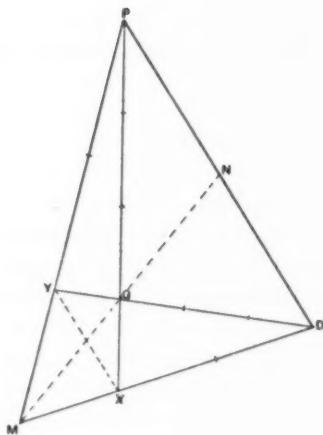


FIG. 2

pyramids, for all pyramids may be considered composed of triangular pyramids standing on the same basal plane. Leonardo's "axis" here extends from the center of gravity of the base of the pyramid to the vertex.

In connection of the above result of a  $\frac{1}{3}$  of the way up the median leading to a  $\frac{1}{4}$  of the way up the axis it may be noted that this is a special case of a more general theorem for triangles. It can be shown, as above, that if two points  $X$  and  $Y$  divide

the two sides  $MD$ ,  $MP$  of any triangle  $PMD$  such that  $MX/MD=MY/MP=1/n$  then will  $XG/XP=YG/YD=1/(n+1)$ . Thus  $\frac{1}{4}$  leads to  $\frac{1}{3}$ ,  $\frac{1}{2}$  to  $\frac{1}{6}$  and so on. Possibly Leonardo knew this.

It is also worth noting that if, in all cases, the line  $MG$  is produced to cut  $PD$  in  $N$ , the point  $N$  bisects  $PD$ , i.e.  $MN$  is a median of the triangle  $PMD$ . This follows from Ceva's Theorem which states for any three concurrent lines drawn from the vertices of a triangle to their opposite bases respectively the product of the alternate segments  $MX$ ,  $DN$ ,  $PY$  equals the product of the other alternate segments  $XD$ ,  $NP$ ,  $YM$ . This expression becomes, in our case,

$$\left(\frac{1}{n} MD\right)(DN)\left(\frac{n-1}{n} PM\right) \\ = \left(\frac{n-1}{n} MD\right)(NP)\left(\frac{1}{n} PM\right)$$

whence  $DN=NP$ . Also  $(n-1)MG=2GN$  and therefore in Leonardo's diagram  $MG=GN$ . We may also say, "The center of gravity of a tetrahedron bisects the line joining the mid-points of two opposite edges of the tetrahedron."

### The Decimal Point

(Continued from page 575)

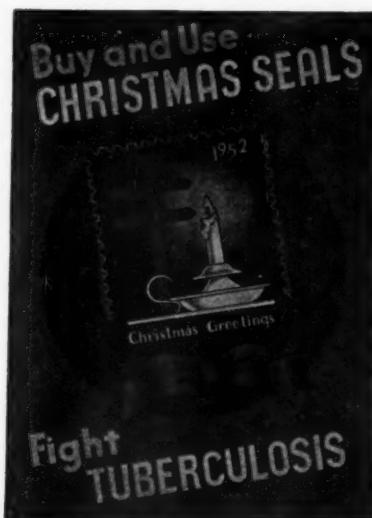
$M_A$  as required in the right member of (4). Hence,

$$M_Q = (M_A + 1) - M_B. \quad (7)$$

Or,  $M_A = M_Q - (1 - M_B)$ ; see Figure 5. Figure 5 shows a division of  $A$  by  $B$ , and reveals that the slide must extend to the left in order for  $M_A < M_B$ . Subtracting (7) from (4),

$$C_Q = C_A - (1 + C_B).$$

Thus, when the slide extends to the left, the characteristic of the quotient is equal to the characteristic of the numerator minus the characteristic of the denominator plus unity.



## An Illustration of the Common Statistical Averages

By ARTHUR E. HALLERBERG  
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IN ORDER to illustrate the use of some of the more common measures of central tendency in the field of statistics, it is usually necessary to refer to frequency distributions based upon different kinds of observations or measurements. This is natural and appropriate, since such measures as the arithmetic mean, the mode, and the median, do not always give equally good representative values for a given set of numerical data. Different types of frequency distributions (i.e., groupings of data into classes with the corresponding frequency of each class), or at least different distributions of the same general type, must therefore be used to illustrate these statistical averages and to show when each gives a more typical value.

It is possible, however, to illustrate these three measures by referring to just one specific frequency distribution, and to obtain meaningful although different results in each case. We refer here to the type of distribution embodied in the so-called *life table*, perhaps more commonly referred to as a *mortality table*. In its simplest form, such a life table may be thought of as giving the "life" (or "death") history of a certain large number of persons, for example, 100,000, born at the same time into some well defined category (national, racial, geographical, or the like). In effect, records are kept on this particular group as to the age at which each individual dies. These data can then be used to determine the number of persons of the original group dying at a particular age, the number still alive at any given age, etc.

Such a table would probably require almost a hundred years for its completion; by that time it would be of little value except for historical purposes. Having no such table at our disposal, we shall there-

fore base our illustration on the Commissioners' 1941 Standard Ordinary Mortality Table. This table was constructed by a different method, but for the purposes of this illustration it may most easily be thought of as representing the above situation.<sup>1</sup> Since 1948 this table has widely replaced the American Experience Table of Mortality in many of the computations involved in various forms of life insurance.

Each of the three measures mentioned above gives a particular type of "average" or "representative value" for a set of data. We may therefore apply these concepts in using a mortality table to find some sort of answer to such questions as "how long will the 'average' person live," "what is the 'average' length of life," etc. It should not be necessary here to point out that nothing can really be said about predicting the length of life of any one individual person; we can only use the experience of a large group of persons in predicting what will happen to another large group. However, it is interesting to set up some sort of hypothetical "average" person and see what might be said about how long, "on the average," he will live. Here one of the important results is that the

<sup>1</sup> Obviously this is a decided oversimplification of the matter, but a detailed account of what is actually represented by such a table would defeat the purpose of this illustration. Essentially this table was prepared from the mortality records of life insurance companies during the ten year period from 1930 to 1940. For details of how a life table may be constructed, and for further references on the Commissioners' 1941 Table, the reader is referred to the informative and very readable book by L. I. Dublin, A. J. Lotka, and M. Spiegelman, *Length of Life, a Study of the Life Table*, Ronald Press, revised edition, 1949, pages 10-25, 294, 303-332. This book also contains many useful and interesting comparative figures, tables, and charts on the different causes of death for various groups classified according to age, sex, locality, nationality, and the like.

answer depends on exactly how the question is stated. And this is where the application of the different statistical averages enters into the picture.

Suppose we consider the Mortality Table as a frequency distribution with the age at which death occurs as the *variate*. (In this case the variate also represents the number of whole years that each individual lives.) The length of the *class interval* is one year. *Frequency* therefore represents the number of persons (from among the original group) who die in the year of stated age, that is, in the year following a given birthday. As usual, it will be assumed that the deaths occur uniformly throughout the year.

The *median* is defined to be that value above which and below which lies an equal number of variates; hence it is often referred to as the "middle" score. (If there is an even number of variates, or scores, the median is usually defined as the value midway between the two middle scores.) Using the extended form of the table, we find that of the original 1,023,102 persons who might be considered as forming our hypothetical group, only 506,403, slightly less than half, attain the age of 68. Hence the median age of death for the entire group, that is, that age which 50 per cent of the entire group attains and the other 50 per cent does not attain, is, by interpolation, a little less than 67.8 years.

This value of course changes if we consider the median of a particular group which has already attained a certain age. For example, half of the 927,763 who attain the age of 29 are still alive at the age of 69.6 years. Hence, as would be expected, the median age of death of the group which has actually attained age 29 is greater than the median of the original group as a whole.

It will be recalled that the *mode*, in its more elementary form, is defined as the "most popular" score—the variate occurring with the greatest frequency. (For many frequency distributions a more re-

fined definition of the mode is necessary, but the above is adequate for our purpose.) It is a simple procedure to check the number of deaths occurring in each year and to find that the number of deaths during the year of age 74 is 28,154, the largest number of the original group to die at any particular age.

We finally examine the most important average, the *arithmetic mean*. This is defined to be the sum of the variates divided by the number of variates. It is only by reference to the arithmetic mean that the often used (and sometimes misused) expression, "the expectation of life," can be fully appreciated. Actually, the expression might more appropriately be replaced by "average future lifetime," since intuitively the idea of expectation of life might be confused with either of the two previous concepts.

By methods to be described below, the average future lifetime (the complete<sup>2</sup> expectation of life) of an individual at birth according to the Commissioners' 1941 Table is 62.33 years. At age 29, the average future lifetime is 38.61 years.

With the idea of the arithmetic mean in mind, the average future lifetime for any given age is practically self-explanatory. It must first of all be noted that the expectation of life must always be given with respect to some particular age. Thus, to find the average number of years still to be lived per person who has reached the age of 29, it is necessary to know the total number of years lived beyond age 29 by all persons of our original group who attained the age 29. The average future lifetime of persons in this group is then found by dividing this total number of years lived by the number of persons attaining age 29. Except for the very highest ages, the computation of the average future

<sup>2</sup> The "complete" expectation of life is here distinguished from the "curtate" expectation of life, which gives the average number of whole years lived after attaining a given age. The curtate expectation would therefore be about one-half year less than the complete expectation figure.

lifetime is somewhat lengthy. Using the data available in this table, the expectation for age 96 could be computed in the following manner:

If one assumes that the deaths occur uniformly throughout the year, the average time lived per person after his last birthday would be one half year. Now let us consider the 1818 persons who reach their 96th birthday. The 813 persons dying during the year after the 96th birthday contribute 406.5 years to the total number of years still to be lived by this group; the remaining 1005 persons contribute one year each, until their 97th birthday. Repeating this process for ages 97, 98, and 99 gives us the following results:

the experience of the Mutual Life Insurance Company of New York. Since it represented a conservative standard, it was almost universally used in this country for calculating policy reserves on standard ordinary insurance and for the valuation of insurance contracts. As there has been considerable improvement in the mortality rate since the older table was formulated, especially in the younger ages, it was felt that more up to date tables should be used. Since January of 1948 the American Experience Table has been replaced by the Commissioners' 1941 Standard Mortality Table.

The table on the next page gives a comparison of the values obtained by the

Age	Number attaining stated age	Number dying during year	Total number of years lived by those dying during year	living through year
96	1818	813	406.5	1005
97	1005	551	275.5	454
98	454	329	164.5	125
99	125	125	62.5	000
			909	+ 1584 = 2493

Adding the last two columns together therefore gives, for the group of persons actually attaining age 96, the total number of years lived by them after they attain that age. The average future lifetime, computed as an arithmetic mean, is therefore  $2493/1818$ , which gives 1.37 as the complete expectation of life for age 96.

A nationally syndicated press release early in 1952 stated: "A noted educator predicts that within the next quarter century the average life span will be 100 to 120 years." Now just what the educator (or possibly the reporter) meant by "average life span" is of course difficult (if not impossible) to say. A comparison of the values mentioned above obtained from the Commissioners' 1941 Table with its main forerunner, the American Experience Table, may be of interest in attempting to evaluate such a conjecture.

We may first of all recall that the American Experience Table dates back to the year 1868, and was primarily based upon

methods above, using the two tables. Figures in parentheses give the sum of the expectation of life at the stated age plus the stated age—they therefore represent the average length of life (in the sense of the arithmetic mean) for all those attaining the stated age.

These figures show very vividly the result of advances in prenatal, birth, and early childhood conditions in more recent years. They also make one wonder what as yet unannounced elixir of life is in store for us which will raise the "average life span" within the next quarter of a century to 100 to 120 years. (More correctly, perhaps, they make us wonder what new statistical average has been invented which would yield such results.)

It has already been pointed out that the results of these two tables are applicable only to a large group. Actually we must add that they are applicable to a special large group, namely, to those who are considered "good insurance risks." This

	American Experience Table (1868)	Commissioners' Standard (1941)
Median (for the entire group at birth)	47.7	67.8
Mode	73*	74
Complete Expectation of life at age:		
0	41.45 (41.45)	62.33 (62.33)
10	48.72 (58.72)	55.47 (65.47)
29	36.03 (65.03)	38.61 (67.61)
60	14.10 (74.10)	14.50 (74.50)
80	4.39 (84.39)	5.06 (85.06)
90	1.42 (91.42)	2.58 (92.58)

\* The number dying at age 74 was almost exactly the same as the number dying at age 73.

includes such factors as being qualified physically, financially, and the like, and hence each table would not pertain to a random selection of individuals. The more general life tables, although used in the early history of life insurance, have thus been replaced by tables which endeavor to fit only the particular group to which they are applied.

In conclusion, it might be well to summarize how such a mortality table can be used as a teaching and enrichment aid:

1) In the form of a single frequency distribution it allows the illustration and application of the three most common statistical averages in a decidedly meaningful situation. In connection with the expectation of life, it also illustrates how an arithmetic mean may be increased without appreciably increasing the size of the largest variates.

2) It vividly illustrates the role of *definition* in a life situation as well as in mathematics. The student should see how different answers can be given to the question: "What is the 'average' length of life?" depending upon the particular meaning which is attached to this question.

3) It permits a pointing up of the role of *assumptions* in our thinking. A helpful exercise for the student would be to list some of the assumptions involved in using any of these tables. For example, the customary assumption that the variates are evenly distributed throughout the class interval (hence that the deaths are uniformly spaced throughout the year) is usually considered to hold for this type of distribution, except for the first year after birth. For the first year the deaths occur in such a way that the year must be broken up by months in order to present the true picture. According to one life table for white males, the death rate per 1000 varies from 31.05 the first month, to 3.59 the second month, to only 0.70 the twelfth month.<sup>3</sup>

Finally (although this aspect has not not been considered in this paper), the mortality table can also be used to illustrate some of the applications of *probability* and other aspects of mathematics to the actuarial field, which some of our ablest students in mathematics may well be encouraged to enter.

\* *Ibid.*, page 24.

The National Teacher Examinations will be given at 200 testing centers throughout the United States on Saturday, February 14, 1953. At the one-day testing session a candidate may take the Common Examinations, which include tests in Professional Information, General Culture, English Expression, and Non-Verbal Reasoning, and one or two of eight Optional Examinations designed to demonstrate mastery of subject matter to be taught. For application forms and further information write: National Teacher Examinations, Educational Testing Service, Princeton, New Jersey.

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## WHAT IS GOING ON IN YOUR SCHOOL?

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and

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### THE MULTI-CONVERSE CONCEPT IN GEOMETRY<sup>1</sup>

Suppose a theorem has  $m$  conditions in the hypothesis and  $m$  conditions in the conclusion. According to Lazar<sup>2</sup> a converse may be obtained by interchanging any number of conditions in the conclusion with an equal number of conditions in the hypothesis. Our theorem would then have  $C_m^{m+m} - 1$  apparent converses although these may not all be distinct after they are verbalized.

It is the purpose of this paper to elaborate on Lazar's thesis to the extent of showing that certain well-defined pedagogical advantages can result from the systematic classroom use of the multi-

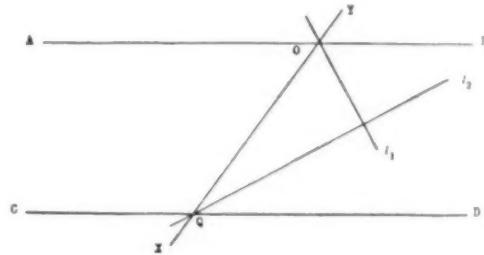
<sup>1</sup> Department Editor's Note: Some plane geometry teaching is too formal, following tradition, and leaves little opportunity for originality on the part of the student. In Mr. Allen's account of his use of the multi-converse concept in geometry, it is shown how students can be given an opportunity to state new theorems, some false as well as some true, and to prove them, false or true as the case may be. This is in keeping with best current trends in teaching mathematics at all levels, and is in the opinion of the editors of this Department an excellent illustration of the "discovery" approach in geometry.

Because of the importance of these concepts, often neglected in geometry, and because of the ever greater importance of Mr. Allen's way of taking care of them in his classes, the editors feel that this account makes a fine contribution to materials on what is going on in our schools. The content of this paper was recently presented at a National Council meeting and Mr. Allen was asked to send selections from his address to this Department so that other teachers could see what he is doing in one phase of his geometry course at Lyons Township High School.

<sup>2</sup> Nathan Lazar, "The Importance of Certain Concepts and Laws of Logic for the Study of Teaching of Plane Geometry," THE MATHEMATICS TEACHER, XXXI (1938) 99-113; 156-174; 216-240.

converse concept in the study of geometry. Several examples will be presented showing how the concept can be applied to familiar geometric situations and then the supposed advantages will be offered for consideration.

*Example 1.* The bisectors of the consecutive interior angles formed when two parallel lines are cut by a transversal are perpendicular to each other.



**Hypothesis:** Transversal  $XY$  intersects lines  $AB$  and  $CD$  in points  $O$  and  $Q$  forming consecutive interior angles  $QOB$  and  $OQD$ ;  $l_1$  passes through  $O$  and  $l_2$  passes through  $Q$ . Also

- (1)  $AB \parallel CD$
- (2)  $l_1$  bisects  $\angle QOB$
- (3)  $l_2$  bisects  $\angle OQD$

#### Conclusion:

$$l_1 \perp l_2.$$

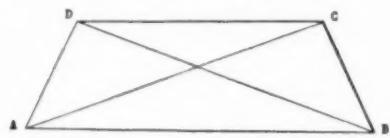
The class is asked to form the three converses obtained by exchanging the conclusion with each of the numbered items in the hypothesis. They are then required to devise a statement in the if-then form to describe each of the geometric situations thus obtained. They see that the same statement results from the last two exchanges so that only two of the converses are distinct after they are verbalized.

These two may be expressed as follows:

1. If the bisectors of two of the consecutive interior angles formed when two straight lines are cut by a transversal are perpendicular to each other then the lines are parallel.
2. If one of two consecutive interior angles formed when parallel lines are cut by a transversal is bisected and a line is drawn perpendicular to this bisector from the vertex of the other angle then this line bisects the other angle.

The students know of course that a converse of a true statement is not necessarily true. The question of the validity of these two statements must therefore be thoroughly investigated. If a student believes that a converse is false he may prove this by showing that it is inconsistent with previously proved theorems. Or he may rest his case on a single exception as shown in a diagram. If he believes it to be true he is required to present a proof. In this case it is found that both of the converses are true and that direct proofs may be obtained for them.

*Example 2.* If a quadrilateral has equal diagonals and one pair of opposite sides parallel, then the other pair of opposite sides are equal and they make equal angles with each of the parallel sides.



**Hypothesis:**  $ABCD$  is a quadrilateral having diagonals  $AC$  and  $BD$  and

$$\begin{aligned} H_1 \quad & AC = BD \\ H_2 \quad & DC \parallel AB \end{aligned}$$

**Conclusion:**

$$\begin{aligned} C_1 \quad & AD = BC \\ C_2 \quad & \angle DAB = \angle ABC \\ C_3 \quad & \angle ADC = \angle DCB \end{aligned}$$

After this is proved it is seen that there are nine apparent converses. However the same statement sometimes results from two different exchanges so that only six of these converses are distinct after they are verbalized. These six can be stated as follows.

1. ( $H_1C_1 \rightarrow H_2C_2C_3$ .) If a quadrilateral has equal diagonals and one pair of opposite sides equal, then these equal sides make equal angles with each of the other two sides and the other two sides are parallel.

2. ( $H_1C_2 \rightarrow H_2C_1C_2$  and  $H_1C_3 \rightarrow H_2C_1C_2$ .) If a quadrilateral has equal diagonals and one pair of opposite sides make equal angles with a third side, then this pair of opposite sides are equal, the other pair of opposite sides are parallel and the other two angles of the quadrilateral are equal.

3. ( $H_2C_1 \rightarrow H_1C_2C_3$ .) If a quadrilateral has one pair of opposite sides parallel and the other pair equal, then these equal sides make equal angles with each of the parallel sides and the diagonals are equal.

4. ( $H_2C_2 \rightarrow H_1C_1C_3$  and  $H_2C_3 \rightarrow H_1C_1C_2$ .) If one pair of opposite sides of a quadrilateral are parallel and the other pair of opposite sides make equal angles with one of them, then this latter pair of opposite sides are equal, the other two angles of the quadrilateral are equal and the diagonals are equal.

5. ( $C_1C_2 \rightarrow H_1H_2C_3$ .) If two opposite sides of a quadrilateral are equal and make equal angles with a third side, then the other pair of opposite sides are parallel, the other two angles of the quadrilateral are equal and the diagonals are equal.

6. ( $C_1C_3 \rightarrow H_1H_2C_2$  and  $C_2C_3 \rightarrow H_1H_2C_1$ .) If the angles at the extremities of one side of a quadrilateral are equal to each other and the angles at the extremities of the opposite side are also equal to each other, then these two opposite sides are parallel, the other two opposite sides are equal and the diagonals are equal.

Investigation by the class will show that only the first, fourth, fifth, and sixth are true.

*Example 3.* If two perpendicular lines intersect on a circle and one passes through the center then the other is a tangent.

For purposes of this discussion the three converses will be listed without reference to a drawing. They are obtained by the interchange of conditions in the manner shown above.

1. If two lines intersect on a circle and one is a tangent and the other passes through the center then they are perpendicular.
2. If two perpendicular lines intersect on a circle and one is a tangent, then the other passes through the center.
3. If one of two perpendicular lines passes through the center of a circle and the other is a tangent then they intersect on the circle.

These examples are perhaps sufficient to show how the multi-converse concept can be applied and to give some idea of its scope. The following advantages of the method are offered for consideration:

1. It provides intensive training in the precise and accurate use of English. The effort to express the proposition indicated by the symbols in terms of a general statement invariably raises class interest to a high level. The students learn that mere fluency is not enough, that words must be used with careful attention to their meaning. Usually a good many efforts are made before someone succeeds in framing a statement that is acceptable to the entire class. Even then others will try to improve on this statement so that the final form will be smooth and more elegant. Needless to say, it is frequently necessary to appeal to some of the fundamental rules of grammatical construction.
2. It is a method of discovery. The student can devise his own theorems and exercises by means of these simple manipulations of the data in the hypothesis and conclusion. When he considers each new statement he asks, "Is this true?" This is psychologically quite different from being confronted with a new theorem whose proof he is told to master. This constant questioning of the validity of new generalizations creates an atmosphere which fosters an inquiring, contemplative frame of mind.
3. It traces a strong logical connection between certain sets of theorems. The student is encouraged to see theorems in groups rather than singly. He is equipped with a method whereby, given a key theorem, he can obtain all the theorems in a mutually converse set.
4. The method can be used to develop an almost inexhaustible supply of highly pertinent exercises.
5. The method may take the teacher, as well as the pupil, off the beaten track occasionally.
6. It stresses repeatedly the fact that a converse of a true statement is not necessarily

true and so puts the student on guard against the many insidious forms of that fallacy.

7. It stresses the fact that the validity of a general statement can be destroyed by citing a single exception.

#### Contributed by

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#### HOME-MADE PROBLEMS FOR ALGEBRA

The list of problems given in this article was compiled by students in a course in the Teaching of Secondary Mathematics given at Western Michigan College last fall. It is submitted with their permission in the hope that it may suggest an activity which some teachers will find helpful.<sup>3</sup>

The idea of making such a list originated while we were considering difficulties which ninth grade students have in connection with verbal problems in algebra. The opinion was frequently expressed that many of the verbal problems found in textbooks are such that ninth grade students look upon them as being neither useful nor very interesting. The recurrent expression of this opinion led to the suggestion that students might try making up some problems of their own, the idea being that they would find such problems more interesting than book problems, and thus would work at them with more zest. Out of this grew the proposal that the members of our class, as prospective teachers of mathematics, try their own hand at this, and the list of problems presented herewith resulted from that undertaking.

The 23 members of our class, all juniors or seniors in college, decided that each would make up and submit one or two original verbal problems which he or she considered interesting and suitable for use in ninth grade algebra. This was done and between 30 and 35 problems (presumably original) were submitted. These were then turned over to a commit-

<sup>3</sup> Unfortunately, limitations of space make it impossible to list the names of the 23 college students who participated in this exercise. Credit for the article really belongs to them.

tee of three members of the class for appraisal. The 20 problems listed below are the 20 which this committee selected as the best of the ones turned in. They are given here without change and without further comment.

The members of the class felt that this little project was worthwhile though it was not as easy as they had thought it would be. They felt that ninth grade students, with a little help, might very well make up some verbal problems of their own and that this could add interest and understanding to this part of their work in algebra.

#### THE 20 PROBLEMS

1. In a football game three backfield men threw 30 passes. The first threw 3 times as many as the third, and the second threw  $\frac{1}{2}$  as many as the first. How many did each throw?

2. In the finals of the State Basketball Tournament, Bonville won 52 to 38. Bonville's forward line of Peters, Jackson, and Belden scored 34 of these points. If Jackson scored 8 more than Peters, and Belden scored 12 more than Jackson, how many points did each score?

3. Peter Piper picked a peck of pickled peppers in six minutes. The first three minutes Peter Piper picked twice as many pickled peppers as he did the last three minutes. If there are 45 pickled peppers in Peter Piper's Peck, how many pickled peppers did Peter Piper pick in each three minutes?

4. One year a baseball player got 175 hits, some of which were singles, some doubles, some triples, and some home runs. He hit one more double than he did home runs, one less than half as many triples as home runs, and 15 more than four times as many singles as doubles. How many of each did he hit?

5. Two-thirds of the fleas on a dog are killed after the first application of flea powder. The second application of powder kills one half of the remaining fleas. If the dog has 35 live fleas after the second application, how many fleas did it have originally?

6. If a mule takes one keg of wine from a donkey's back, he is carrying three times as much as the donkey, but if he gives the donkey one keg of his own, the mule is carrying a load equal to the donkey's. How many kegs does each have in his original load?

7. The combined ages of a family of five persons total 145 years. Father is 8 years older than Mother. Mother is one year less than twice as old as Mary. The combined ages of the two boys total 34 years and one is four years older than the other. Find the ages of the individual members of the family.

8. The Rochester football team scored a total of 158 points during the season. Allen

scored only by kicking extra points. The rest of the scoring was done by Miller, Jones and Woodruff. If Jones scored 2 more touchdowns than Miller, and Woodruff scored 10 more touchdowns than Jones, and Allen missed only 3 extra points during the season, how many points did each man score?

NOTE: A touchdown counts 6 points, and an extra point is 1.

9. A boy with 20 cents to spend went into the candy store to buy some gum and some candy. He wished to spend three times as much for candy as for gum. How much candy and how much gum could he buy with his 20 cents if gum is 5 cents a package and candy costs 5 cents a bar?

10. If an oil drum can be filled by one tap in 9 hours, by another tap in  $4\frac{1}{2}$  hours, and by still another tap in 1 hour, how long will it take to fill the drum if all three taps are working at once?

11. Tom, Dick, and Rudy went fishing for bass. Tom caught three times as many fish as Dick, and four times as many as Rudy. Their total catch was 38 fish. How many fish did each boy catch if the largest fish was 17 inches long and weighed  $3\frac{1}{2}$  pounds?

12. If you were playing records on an automatic record player which would play 10 records without attention, and you found that each record averaged 2 minutes 40 seconds to play, and it took 7 seconds for each record to change how many records could you play in one hour? Allow 2 minutes to change the records.

13. Dick hit a softball  $70x$  feet and the ball went on a direct line from home plate over second base. Second base is 120 feet from home plate by way of first base. How far beyond second base did the ball go if  $x$  equals 3? Find the computed distance from home plate to second base correct to the nearest foot.

14. This problem concerns a rootin' tootin' bandit-catching sheriff, and has nothing to do with the number of shots a six-shooter really holds. The sheriff is chasing a bandit and shoots  $\frac{1}{2}$  of his supply of bullets during the first mile. He shoots five less during the second mile, five less than that during the third mile, and five less than that during the fourth mile. At the end of four miles he finds that he has shot all his bullets without effect, and he is forced to catch the bandit with a lasso. How many bullets did he have when he started?

15. Jack and Bill went fishing. Bill caught 4 bass, 3 perch, 5 sunfish, and several others. Jack's luck wasn't so good. He had as many bites as Bill had, but landed only  $\frac{1}{2}$  as many fish. On the way home the fish became heavy, so the boys sat down to rest. After taking a short nap, they found that a stray cat had eaten up  $\frac{1}{3}$  of their total catch. How many fish did each boy catch if they now have 20 fishleft?

16. The school Jim attends is located on top of a hill. There are three flights of steps to the school building, and Jim wants to know which flight has the fewest steps. The front stairs have  $\frac{1}{2}$  as many steps as the back stairs, and the side stairs have 10 fewer steps than the back stairs. If there are 210 steps altogether, which

flight is the shortest, and by how many steps is it shorter than each of the others?

17. At a class meeting consisting of 96 students there had to be a vote to see if the students wanted to go on a class picnic. Under normal circumstances, everyone would like this entertainment, but since there were a few "pantywaists" and "bug haters" in the class, the motion was carried by only 24 votes. If each student were to eat three hotdogs at a cost of 10 cents each, how many students voted in favor of a picnic?

18. Paul, although pleased with the large vote in today's council election, now has to see whether any candidate has won a majority of the votes for president. He knows that Jack received 155 votes; Ted, 106; Al, 57; and Tom, 8 votes. Ken tells him "I can total these votes without adding." How does Ken do this? Has any one of the boys won a majority? HINT: Notice that each number of votes is a common number ( $n$ ) less than the next. If  $155 = a$  and  $8 = b$ , derive a formula for the total number of votes ( $s$ ) in terms of  $a$  and  $b$ .

19. If it takes Mortimer 20 minutes to walk the mile from his home to school, and if it take Bessie 15 minutes to talk the half-mile from her home to school, how long will it take for Mortimer to get to school if he picks up Bessie half way from her home to school?

20. The total cost (car expenses) of a 5-day trip from Kalamazoo to San Francisco was \$52.50. If 2 quarts of oil were needed a day and the cost of gas per gallon is  $\frac{1}{2}$  of the cost of a quart of oil, and if the number of gallons of gas used each day is equal to the number of cents which a quart of oil costs, how many gallons of gas were used on the trip?

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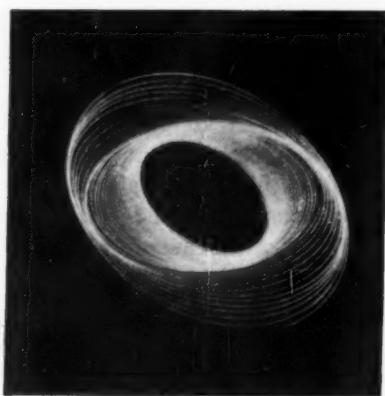
#### LIGHT LOCUS

Invariably the plane geometry class reaches the unit on locus. I begin to explain, "Locus is the path a point traces as it moves . . ." I carefully look at the students, one by one, from the dullest to the brightest, only to be met with blank expressions. Vapor trails left by jet planes, ski trails in the snow, the new-mown path of grass behind the mower, the water in the fountain, the tip of a moving propeller, fireworks, dotted lines in comic strips to indicate motion, and numerous other examples are mentioned. A glimmer of recognition begins to appear on a face or two. At this point one of the students blurts out, "It's always an imaginary line!"

This year at this point, the class and I began looking for illustrations of locus. Many pictures were brought in. The most effective were the ones of patterns traced by lights on moving objects: the light pictures drawn by Pablo Picasso which appeared in *Life*; the intricate and beautiful patterns made by Skater Carol Lynne with flashlights on her toes and hands, which also appeared in *Life*; traffic patterns of lights at night; the air traffic patterns in busy city airports; tracer bullets; the patterns made by lights on twirling batons. One and one, the illustrations came: motion study made possible by time exposures, the effectiveness of springs in automobiles as they were driven over rough roads, and the use of lights to study the precision of the movements of ballet dancers. But all of these were light paths made and recorded by other people.

Then one morning before eight o'clock in rushed David Au, an avid photography fan, with a copy of *Modern Photography*, November, 1949. On the cover were some unusual geometric curves. These were used to attract attention to the article, "Pendulum Patterns," page 70, by Donald Nusbaum. Together David and I read the article. The next morning David brought in some lovely pendulum patterns he had made with his camera and a flashlight. And these light paths were some that David, one of us, had made.

To make pendulum patterns, a flashlight suspended from the ceiling is used as the pendulum. The camera is placed on the floor directly below the flashlight and is properly focused to bring the light source into focus. The light source must not be too large. David, in making his patterns, used two batteries with the bulb taped to them and a sheet of black paper behind the bulb to absorb the excess light. Tommy Taylor, another student, removed the lens of the flashlight and blacked the reflector with black shoe polish to prevent too great a dispersion of light. Another group simply took several thicknesses of black construction paper, cut a tiny hole in the center and put this over the lens.



Pendulum pattern made by David Au.

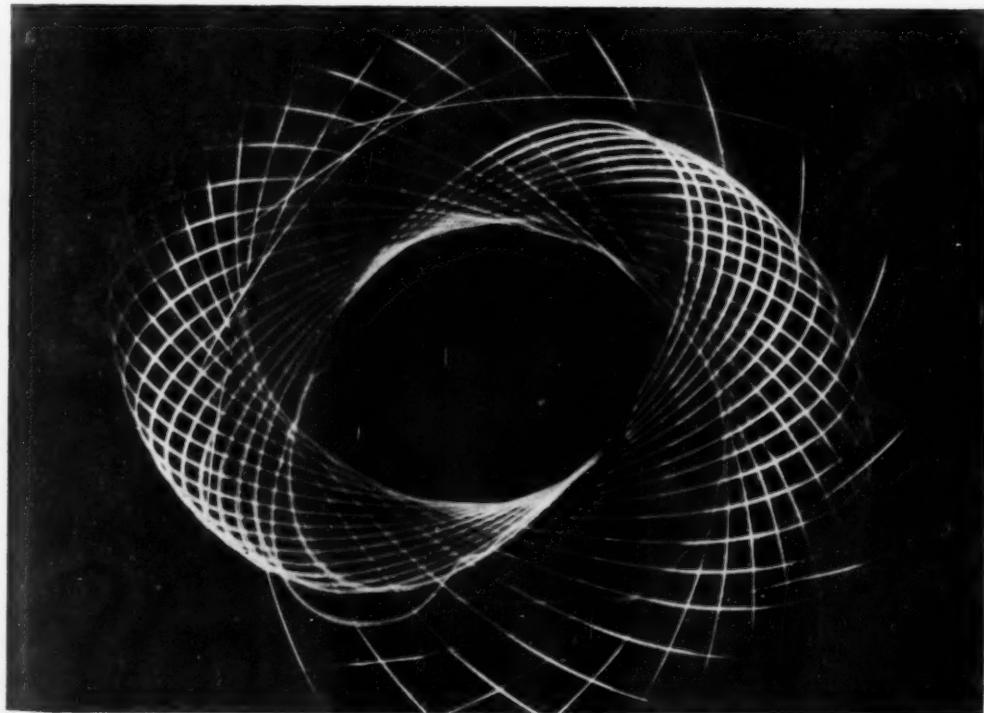
In taking the picture, the lens of the camera was opened as wide as possible in most cases. After the flashlight settled into an even swing, the shutter of the camera was opened. The length of the exposure depended on the swinging of the pendulum.

There is no limit to the variety of effects possible. Double exposures produce interesting effects. A variation is to have the pendulum hang from a "Y" instead

of a single string. The use of colored filters gives beautiful color patterns.

Much ingenuity was shown by the pupils in working out their patterns. These were light paths which they themselves worked out. They saw the locus in the "making" and the camera did for them what they could not do—recorded the locus. Always with amazement and wonder we viewed the "light locus"—amazement at its beauty and wonder that we had "made" a locus.

Howard Fisackerly recorded an excellent cycloid as a result of a problem in our text and of the incredulity of the students that such a path would be made by a point on the circumference of a circle as the circle moved along a straight line. Other geometric curves could be made. In fact, the possibilities are endless. Pendulum patterns correlate with photography and physics and illustrate clearly the meaning of locus. The camera records the path the moving point traces as it follows certain conditions. No longer do students ask, "Is it an imaginary line?" There recorded



Pendulum pattern made by Cleon Bowers.

in black and white or in color is the path on the film for the student to see.

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#### MATHEMATICS CURRICULUM STUDY AT ELGIN

The Mathematics Curriculum Committee in the Elgin, Illinois school system was organized three years ago in response to the teachers' request for a better knowledge and understanding of the work offered at different grade levels than their own.

With the realization that any good program in mathematics, with the emphasis on the progressive development of the child, must have a planned sequence beginning with the kindergarten and extending through the fourteenth grade, a committee of nineteen members was formed. These members were selected in such a way that all buildings and all grades were represented by one or more members. This made it possible to keep each building informed on the work as it progressed. The Committee was headed by two co-chairmen, the head of the high school mathematics department, and a grade school principal. There were four subcommittees; primary, including kindergarten through third grade; intermediate, fourth through sixth grade; junior high, seventh through ninth grade; and senior high. Each subcommittee elected its own chairman.

The committee, believing that there must be (1) minimum essentials of mathematics necessary for responsible citizenship, (2) a sequential program for development and maintenance of skills, and (3) a provision for achieving functional competence in mathematics, had as its aim last year the examination of the experiences provided in mathematics at various grade levels, the placement of them in sequential pattern, and the addition of teaching vehicles for many of these experiences.

Upon examination of the past experiences, the conclusions of the committee were:

1. Fewer broad topics should be attempted so that a greater degree of mastery might be attained.
2. The program should be unified in order to avoid necessary repetition.
3. Meaning and understanding of the arithmetical processes should be given by identifying the needs, experiences, and activities of boys and girls at various grade levels.

Each subcommittee worked separately at first gathering suggestions and materials from all teachers in the system. The subcommittees then came together and worked as an entire committee for unification of the program. Consultant service was secured from the Illinois Curriculum Program. This service proved helpful to the committee as it worked on specific problems.

Last fall the committee placed in the hands of all teachers a tentative experimental, mimeographed outline (including teaching vehicles) of mathematics from the kindergarten through the ninth grade. It was suggested that the outline be followed this year, rather than any text, so as to discover in what areas the outline is workable and in what areas it needs further revision. The committee also asked that additional teaching vehicles be added as the teachers worked with their groups this year. Since there is no one perfect book available, the committee emphasized the importance of having available in each of the junior high school mathematics classrooms several sets of books to be used as texts.

Since this type of classroom teaching requires many concrete manipulative materials of all kinds and many resources in the classroom for the help of the students, the committee is now investigating new materials which will best fit the program as presented.

The senior high committee has completed the outline of courses offered and

is now in the process of including teaching vehicles.

The fact that teachers of all grade levels have been willing and interested enough to sit around a table to discuss their common problems, and then attempt to do something to improve them is one of the outstanding features of the mathematics curriculum work in the Elgin school system.

From *The Illinois Council of Teachers of Mathematics*, May, 1952, by HORTENSE WILSON, Co-Chairman of the Committee of Elgin Public Schools, Elgin, Illinois

#### A MATHEMATICS CURRICULUM FOR A SMALL HIGH SCHOOL

This discussion deals specifically with the small high school with only one (and possibly only three-fifths of one) teacher in mathematics, with a crowded schedule, and with a small number of students.

*9th Grade.* Elementary algebra of the "traditional" type or the "sequential" type is available each year. All students are required to take either this course or the general mathematics course. Only the above average students (generally speaking, over 100 IQ) are guided into this subject and all of them are strongly advised to enroll.

*10th Grade.* Plane geometry is taken by all those that have had algebra and wish to continue with their mathematics. Strong encouragement is given to all algebra students to go on in geometry. If the students terminate their mathematical study after only one year, they will probably have a very distorted idea as to what mathematics is since they have not had an opportunity to see the geometric side of things. The rights of the individual must be respected, but every effort is made to keep people in at least the two-year program.

*9th and 10th Grade.* For those who are average or below, a course in general mathematics is offered. This course, offered every other year to freshmen and sophomores, consists of a thorough estab-

lishing or re-establishing of the fundamentals of arithmetic, problem-solving, and various topics considered indispensable to living in the twentieth century. It is understood that this course is not a college preparatory course, with the possible exceptions being taken up on an individual basis. With this in mind, the teacher can offer better grades to add to the student encouragement program.

*11th and 12th Grade.* Two different courses, neither a prerequisite for the other, are offered on an every other year basis to juniors and seniors. One of the year programs consists of a semester of algebra, including equations in one unknown of the first and second degree, irrational and imaginary numbers, exponents, logarithms, and ratio, proportion, and variation, and a semester of trigonometry. In the following year (which is the first year for some) material as suggested below is covered.

- A. Review of fundamental operations
- B. The coordinate system
  - 1. Before discussing slope, bring in systems of equations, particularly graphical solutions.
  - 2. Immediately after slope, bring in the distance formula, the equation of a circle, and intersections of lines and circles.
- C. Functional relationships
- D. Quadratic equations with two variables, stressing the graphical approach
- E. Applications to conics
- F. Rates of change
- G. Mathematical induction
- H. Permutations, combinations, and probability
- I. The binomial theorem
- J. Arithmetic and geometric series
- K. The second part of this course is solid geometry including the analytic point of view.

These courses, needless to say, would be for the esoteric few that care to go further in mathematics. Everyone who has completed the first two years of mathematics is encouraged to consider a career in science or mathematics and hence guided into the advanced courses. Not all of the students are so interested, unfortunately.

(Continued on page 593)

## REFERENCES FOR MATHEMATICS TEACHERS

Edited by WILLIAM L. SCHAAF

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### Teacher Education in Mathematics

IN THE 1923 Report on the Reorganization of Mathematics in Secondary Education, it was pointed out that in some states the preparation of high school mathematics teachers was of such a low quality that the Committee judiciously refrained from giving the detailed data on those states. We have come a long way since then. Today, in general, secondary mathematics teachers are professionally and academically as well educated as teachers in any other field.

In my *Bibliography of Mathematical Education*, a monograph published in 1941, but now out of print, there are listed some 80 references on the preparation of mathematics teachers, covering the period roughly from 1920-1940. Some of these older papers are still worth rereading, notably those by W. C. Bagley, A. A. Bennett, W. Cairns, E. Moulton, W. D. Reeve, C. Richtmeyer, I. Turner, and F. L. Wren. Also still timely is the "Report on the Training of Teachers of Mathematics" in the *American Mathematical Monthly* for May 1935, vol. 42, pp. 263-277.

In the last decade or so, conditions affecting teachers of mathematics have again been changing: the internal organization of the high school, the status of teachers in general, the public attitude toward mathematics, the philosophy of teaching mathematics, new channels of in-service training, the increasing literature of mathematics, expansion in the field of applied mathematics—to mention only outstanding factors. It should occasion no surprise, therefore, that there

should appear a revival of interest in the preparation of teachers, as reflected by the recent Symposium on Teacher Education in Mathematics presented jointly by the Mathematical Association of America and the National Council of Teachers of Mathematics at the University of Wisconsin at Madison in August, 1952.

Among the significant trends in the education of mathematics teachers in recent years we may note the following: (1) greater emphasis upon in-service training; (2) increased interest in mathematical workshops, institutes, etc.; (3) greater cooperation and mutual understanding between college teachers of mathematics and secondary teachers; (4) more attention to "background materials" in mathematics; (5) greater familiarity with various areas of applied mathematics; and (6) some insistence upon actual work experience involving the use of mathematics, such as in industry, surveying, engineering, insurance, business, accounting and the like.

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## What Is Going On?

(Continued from page 589)

The teaching schedule for the teacher using this curriculum plan would be as follows:

### FIRST YEAR

1. General mathematics
2. Algebra
3. Geometry
4. Section I of sequence for 11th and 12th grade
5. 8th grade (or other courses)

### SECOND YEAR

1. Algebra
2. Geometry
3. Section II of sequence for 11th and 12th grade
4. 8th grade (optional)
5. Physics (optional)

The main disadvantage here is that there is no continuity between the last two years of the sequential program. Also,

some difficulty is encountered in scheduling all of these courses, particularly the two courses that take people from two classes.

We find this plan helpful for the following reasons: 1) it meets the needs of all of the students in the high school, 2) it provides a course for the most difficult individual to plan for in our school, the below average student, 3) it takes into consideration the teacher's load, which while heavy, is no greater than it would be if carried under another plan, and 4) it makes teaching actually easier and more effective because the classes are relatively small and students are placed according to interests and abilities.

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# DEVICES FOR A MATHEMATICS LABORATORY

Edited by EMIL J. BERGER

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Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. Or if that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

## FRACTION MULTIPLICATION CHART

The fraction multiplication chart suggested here is a near relative of the multiplication chart for integers with which the reader is undoubtedly familiar.

The device consists of two parts—a chart completed as indicated in Figure 1 and a square sheet of plastic with a small square painted in at one corner in some opaque color.

The chart may be constructed on heavy tagboard or artists mounting board with black ink. The large square enclosed within the heavy black lines may be thought

of as representing unity, and the fractions which are to be multiplied appear outside the margins of this large square along its left and bottom edges. Note that the horizontal and vertical lines are extended sufficiently far to separate the different marginal figures.

Two lines are needed on the square plastic sheet. These may be scratched in the surface or painted on with black lacquer. Make them respectively parallel to the top and right edges of the plastic sheet and at distances from these edges equal to the distances between adjacent horizontal or vertical lines on the chart. The little square outlined by the scratched lines and edges of the plastic sheet may be painted red for added effectiveness.

Figure 1 illustrates how the device is used in locating the product of the two marginal fractions  $\frac{2}{3}$  and  $\frac{3}{4}$ . The answer ( $= \frac{1}{2}$ ) appears on the chart beneath the little red square. Note that if the large square included within the heavy lines represents unity then the rectangle outlined by the upper and right edges of the plastic sheet and the left and bottom heavy black lines of the chart is a portion of unity—in this case  $\frac{1}{2}$ . This is precisely the product of  $\frac{2}{3} \times \frac{3}{4}$ . The integers appearing along the left and lower edges are useful in helping students verify this fact and in locating the proper lines to keep in mind when operating the device.

While the chart illustrated in Figure 1 involves only the multiplication of proper fractions whose common denominator is 12, it is perfectly possible to construct charts which will give the products of improper fractions whose common denominator is also 12. This can be accomplished by making the basic square 24

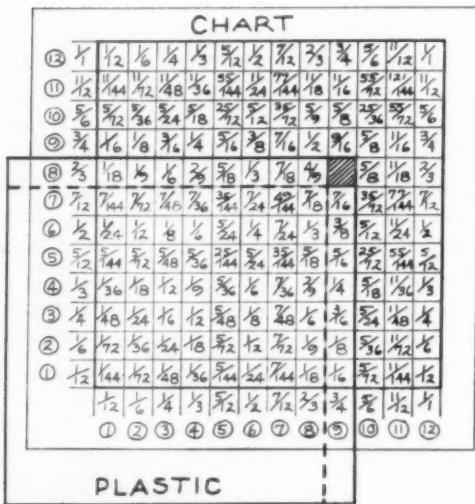


FIG. 1

units on edge, or any higher multiple of 12. Charts involving other common denominators and decimal fractions may be constructed in a similar way.

By using a chart such as this one it is possible to bring home to the pupil quite graphically that the product of two proper fractions can never be as great as unity. When properly constructed and combined with appropriate techniques the chart makes instruction in multiplication of fractions self-teaching to a surprising degree. A little thought will show that the chart can also be used for drill work in division of fractions.

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#### A LOCUS TRACING DEVICE

Nearly every student of plane geometry learns and accepts rather readily the fact that an angle inscribed in a semicircle is a right angle. Almost as many are able to discover that the locus of the vertices of all right angles whose sides pass through two fixed points is a circle with the segment joining the two points as diameter.

Actually the fact concerning the locus is a slight modification of a corollary based on the following general theorem: The locus of a point on one side of a given line segment at which this segment subtends a given angle is an arc of a circle passing through the ends of the given segment.<sup>1</sup>

Consideration of this theorem, possibly following the treatment of the two facts concerning the right angle noted above, will provide a nice opportunity for enrichment. The device suggested in this article will help illustrate the point of view.

Materials needed to produce the device include the following: one piece of plywood  $\frac{3}{8}'' \times 20'' \times 20''$ , two round wooden dowels  $\frac{1}{4}''$  in diameter and 16" long, two small empty spools, one  $\frac{1}{8}''$  stove bolt  $\frac{3}{4}''$

<sup>1</sup> The statement of the theorem is taken from the textbook, *An Introduction to College Geometry*, by E. H. Taylor, and G. C. Bartoo, Macmillan Company, New York, 1949.

long with a wing nut, two  $\frac{1}{4}''$  stove bolts  $\frac{5}{8}''$  long, and two washers for each of the three bolts.

Preparation of the materials involves only a little work with a drill. Drill  $\frac{1}{4}''$  holes in the plywood at A and B (Fig. 2). The distance between these two holes

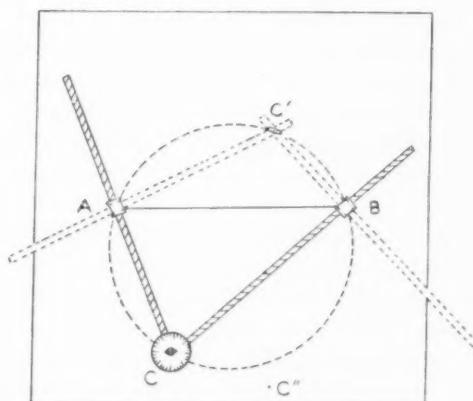


FIG. 2

should be 12". Drill a  $\frac{1}{8}''$  hole in each of the two dowels near one end so they may be joined together as at C in the diagram. Finally drill a  $\frac{3}{16}''$  hole in the side of each of the two spools just far enough to pierce the bore.

To assemble the device begin by fastening the spools to the plywood at A and B as indicated in the diagram. To fasten the spool at A slip a washer over one of the  $\frac{1}{4}''$  stove bolts and pass it through the plywood from the back side. Then turn the bolt into the  $\frac{3}{16}''$  hole in the side of the spool with the aid of a screw driver. The spool at B is fastened in the same way. (Do not attempt to use wood screws; the threaded stove bolts work out much better.) Since the spools must rotate freely when the device is operated, be careful not to draw them down against the plywood too closely.

Next join the dowels together with two small washers and the  $\frac{1}{8}''$  stove bolt and wing nut. A small  $360^\circ$  protractor may be constructed from cardboard and slipped between two washers just under the wing

nut if desired, but this is optional because angle  $ACB$  can always be measured with an ordinary  $180^\circ$  protractor when necessary. Finally pass the free ends of the dowels through the bores in the spools. Line  $AB$  may be painted on the plywood with lacquer.

To fix the size of angle  $ACB$  it is only necessary to tighten the wing nut. When this is done an arc of a circle can be traced out simply by moving the vertex  $C$  from left to right. Variations in the lengths of  $AC$  and  $BC$  needed for the different positions of  $C$  will occur automatically as  $C$  is moved.

The following set of exercises is offered by way of illustration to indicate how the device may be employed in the actual teaching situation.

- (1) Suppose one wishes to find the locus of points on one side of the line segment  $AB$  at which this segment subtends an angle of  $70^\circ$ . To find the locus fix angle  $ACB$  at  $70^\circ$  and trace out  $\widehat{ACB}$  by rotating the vertex  $C$  from left to right. How many arc degrees are there in this arc?
- (2) How can the circle, of which  $\widehat{ACB}$  generated in the preceding example is a portion, be completed? The procedure is not difficult. Loosen the wing nut, pull the joined ends of the dowels up to  $C'$ , set the size of angle  $AC'B$  at  $110^\circ$ , and rotate the vertex  $C'$  from left to right as before. How many arc degrees are there in  $\widehat{AC'B}$ ?
- (3) Find the locus of the vertices of all  $70^\circ$  angles whose sides pass through the points  $A$  and  $B$ .
- (4) Find the locus of the vertices of all  $90^\circ$  angles whose sides pass through the points  $A$  and  $B$ . What is the character of the segment  $AB$  in this case?
- (5) Suppose a line drawn horizontally across the board through  $A$  and  $B$  represents the shore line of a land area located above this line, and that  $A$  and  $B$  are lights on the shore. Suppose also that the area interior to the arc (drawn with angle  $C$  fixed at  $50^\circ$ ) and the shore line is an area of dangerous sailing. If a ship sails in the water (lower half of the

board) along a line parallel to the shore line, how large (or small) must the navigator keep the size of the angle at a hypothetical point  $C''$  in order to avoid danger?

The foregoing list of examples is only suggestive. Once the teacher has the device in his possession he will be able to see other possibilities for problem material. A note of concreteness may be added to tracing exercises by placing a sheet of paper over the plywood and using a pencil to record the movements of  $C$  and  $C'$ .

The device can also be used to help explain what is meant by a cyclic quadrilateral. Place a piece of paper over the plywood, draw lines  $AC$  and  $CB$ , and record the size of angle  $ACB$ . Then swing the joined ends of the dowels across  $AB$  up to  $C'$ , set angle  $AC'B$  equal to the supplement of angle  $ACB$ , and draw lines  $AC'$  and  $C'B$ . The resulting quadrilateral  $ACBC'$  will be cyclic—that is, it can be inscribed in a circle. The circle can be drawn with the device as previously explained.

Several ideas concerning the construction of the device seem worth mentioning. Actually it is possible to produce a similar device entirely of MAKIT TOY. If this suggestion is followed the plywood sheet can be dispensed with simply by substituting a single long rod inserted in two wheels. Anyone familiar with this building set should have no difficulty in constructing the device by using only the materials that come with the set. (See "Models Made From Makit Toy," THE MATHEMATICS TEACHER; Vol. XLIV, pp. 246-247; April, 1951.) As another idea mount each spool on a rubber suction cup and suspend the spools and dowels from a blackboard.

THE MATHEMATICS LABORATORY  
Monroe High School  
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## The President's Page

TWO IMPORTANT meetings were sponsored by The National Council of Teachers of Mathematics in August. In addition to the importance of regular meetings added significance can be attached to these two conferences because each of them was different in several aspects from any previously sponsored by our organization.

A joint meeting of The National Council of Teachers of Mathematics and The New England Institute for Teachers of Mathematics was held on the campus of The Phillips Exeter Academy at Exeter, New Hampshire, August 21-28, 1952. This meeting was the twelfth summer meeting of the National Council, and the fourth summer Institute sponsored by The Association of Teachers of Mathematics of New England, an Affiliated Group of the National Council.

The Exeter meeting was in a sense experimental in that it was the first occasion in which a summer meeting of the National Council had been combined with one of the several successfully operating summer institutes and conferences in the area of mathematics education. It is my personal conclusion that this experiment was a complete success from the point of view of the National Council and I hope also from the point of view of those responsible for the New England Institute for whom of course I can not speak.

The general spirit of the meeting was excellent and the attendance was probably better than would have been true for either separate meeting. For those who would have attended only the summer Council meetings it was very fine to be able to take advantage of the longer period and program annually sponsored by the New England Association. Furthermore, the National Council welcomed this opportunity to give complete endorsement to this summer institute of an Affiliated Group and to the general idea of one-week or two-weeks summer conferences. It is

my hope that in the future other institutes and conferences will invite the National Council to join with them in a similarly planned joint program.

Another element of uniqueness in this meeting is that it was the first meeting of the National Council held on the campus of one of the many fine private secondary schools. It does not seem to me that the facilities, the hospitality, and the efficiency of operation could ever be better than those provided by The Phillips Exeter Academy and the New England Association leaders.

The other August meeting sponsored by the National Council was a Symposium on Teacher Education in Mathematics presented jointly by The Mathematical Association of America and The National Council of Teachers of Mathematics with the cooperation of the School of Education of The University of Wisconsin, and held on the University campus in Madison, August 26-30. The purpose of the Symposium was to consider the college mathematics offerings for teachers of mathematics. This meeting was the first to be devoted to this problem, exclusively, and the first meeting sponsored jointly by these two organizations. The 150 participants included secondary school teachers, mathematics supervisors, school administrators, representatives of liberal arts and teachers colleges and leaders in mathematics research. It is probably the first time in American education that professional subject matter organizations have sponsored a conference for college teachers on content and methods of teaching in subject matter area at the college level.

A more complete report and copies of Symposium recommendations and reports can be obtained by writing to the Director of the Symposium, Professor R. E. Langer, 822 Miami Pass, Madison 5, Wisconsin.

(Continued on page 599)

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## NOTES ON THE HISTORY OF MATHEMATICS

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Edited by VERA SANFORD  
State Teachers College, Oneonta, New York

SIR ISAAC NEWTON ON "HOW A QUESTION MAY BE BROUGHT TO AN EQUATION"\*

"After the Learner has been some time exercised in managing and transforming Equations, Order requires that he should try his Skill in bringing Questions to an Equation. And any Question being proposed, his Skill is particularly required to denote all its Conditions by so many Equations. To do which he must first consider whether the Propositions or Sentences in which it is expressed, be all of them fit to be denoted in Algebraick Terms, just as we express our Conceptions in *Latin* or *Greek* Characters. And if so, (as will happen in Questions conversant about Numbers or abstract Quantities) then let him give Names to both known and unknown Quantities, as far as occasion requires; and express the Sense of the Question in the Analytick Language, if I may so speak. And the Conditions thus translated to Algebraick Terms will give as many Equations as are necessary to solve it.

"As if there are required three Numbers in continual Proportion, whose Sum is 20, and the Sum of their Squares 140; putting  $x$ ,  $y$ , and  $z$  for the Names of the three Numbers sought, the Question will be translated out of the Verbal to the Symbolical Expression as follows:

*The Question in Words*

There are sought three Numbers on these Conditions:

That they shall be continually proportional.

That the Sum shall be 20.

And the Sum of their Squares 140.

"And so the Question is brought to these Equations, *viz.*  $xz = yy$ ,  $x+y+z = 20$ , and  $xx+yy+zz = 140$ , by the Help whereof  $x$ ,  $y$ , and  $z$  are to be found by the Rules delivered above."

[At this point the author shows how the problem may be solved more expeditiously by using the quantities  $x$ ,  $y$  and  $yy/x$ .]

"Take another Example. A certain Merchant encreases his Estate yearly by a third Part, abating 100 *l.* which he spends yearly in his Family; and after three Years, he finds his Estate doubled. *Query*, What was he worth?

"To resolve this, you must know there are or lie hid several Propositions, which are all thus found out and laid down.

*(See Expression on next page)*

"Therefore the Question is brought to this Equation,  $(64x - 14800)/27 = 2x$ , by the Reduction whereof you are to find  $x$ . . . . You see therefore, that to the Solution of Questions which only regard Numbers or the abstracted Relations of Quantities, there is scarce any Thing else

\* Isaac Newton, *Universal Arithmetick*, second English edition, London, 1728, pp. 67-70, 79, and 80. In view of the last problem quoted in this excerpt, it is interesting to note that the lectures on algebra which constitute this volume, were delivered in Cambridge in the Michaelmas term of 1685 at the time when Newton was working on his *Principia* in which the laws of motion were formally stated.

*The same in Symbols*

$x, y, z$

$x:y:y:z$ , or  $xz = yy$ .

$x+y+z = 20$ .

$xx+yy+zz = 140$

*In English*

A Merchant has an Estate

Out of which the first Year he expends 100 *l.*

And augments the rest by one third

And the second Year expends 100 *l.*

And augments the rest by a third

And so the third Year expends 100 *l.*

And by the rest gains likewise one third Part—

And he becomes at length twice as rich as at first.

*Algebraically* $x$  $x - 100$ 

$$x - 100 + \frac{x - 100}{3} \text{ or } \frac{4x - 400}{3}$$

$$\frac{4x - 400}{3} - 100 \text{ or } \frac{4x - 700}{3}$$

$$\frac{4x - 700}{3} + \frac{4x - 700}{9} \text{ or } \frac{16x - 2800}{9}$$

$$\frac{16x - 2800}{9} - 100 \text{ or } \frac{16x - 3700}{9}$$

$$\frac{16x - 3700}{9} + \frac{16x - 3700}{27} \text{ or } \frac{64x - 14800}{27}$$

$$\frac{64x - 14800}{27} = 2x$$

required than that the Problem be translated out of the *English*, or any other Tongue it is proposed in, into the Algebraical Language, that is, into Characters fit to denote Our Conceptions of the Relations of Quantities. But it may sometimes happen, that the Language or the Words wherein the State of the Question is expressed, may seem unfit to be turned into the Algebraical Language; but making Use of a few Changes, and attending to the Sense, rather than the Sound of the Words, the Version will become easy. Thus the Forms of Speech among different Nations have their proper Idioms; which, where they happen, the Translation out of one into another is not to be made literally, but to be determined by the Sense."

At this point, Newton gives a number of verbal problems and their solutions, but having made his point about the representation in algebraic terms, he gives up the parallel column arrangement and

uses the more compact form "Let the least of the numbers be  $x$ ." The problems are concerned with the finding of two numbers given the sum of the numbers as  $a$  and the difference of their squares as  $b$ ; several cases of the problem of pursuit; work problems; mixtures; specific gravity of a mixture, using Hiero's crown as an example; a question involving the present value of an annual pension for five years; and two that are of especial interest the one for its novelty and the other for its connection with Newton's *Laws of Motion*.

These are:

"If the Number of Oxen  $a$  eat up the Meadow  $b$  in the Time  $c$ ; and the Number of Oxen  $d$  eat up as good a Piece of Pasture  $e$  in the Time  $f$ , and the Grass grows uniformly; to find how many Oxen will eat up the like Pasture  $g$  in the time  $h$ ."

"Having given the Magnitudes and Motions of Spherical Bodies perfectly elastick, moving in the same right Line, and striking against one another, to determine their Motions after Reflexion."

**President's Page**

(Continued from page 597)

In order to cover costs of preparation of the reports and mailing, a charge of one dollar is being made.

At the closing session of the Symposium a request was made that the presidents of the Association and the Council take steps to make sure that the two groups continue

the joint study of teacher education in mathematics so well started by the Symposium. While attention has been given to this problem in past committee reports and frequently at National Council meetings, future jointly organized consideration of the problem holds much promise for progress in mathematics education.

JOHN R. MAYOR, *President*

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## MATHEMATICAL RECREATIONS

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*Edited by AARON BAKST*  
185-12 77th Avenue, Flushing 67, N. Y.

WITH THE advent of the holiday season the thoughts of many turn to the selection, or, if it is possible, to the designing of interesting and ingenious messages of greetings. Generally, the prevailing idea that mathematical greeting cards are either dull or uninteresting is far from being correct. A mathematical greeting card may be as attractive as a reproduction of a Currier and Ives or of a Grandma Moses painting. A mathematical greeting card may be a means for some very interesting and instructive activities in a classroom.

During the Medieval times, when medical theory and practice concentrated on the employments of magical means for therapy, much emphasis was placed on the use of amulets. Such objects were worn by the afflicted as means for healing as well as for warding off "evil spirits." One of such amulets was based on the magic word "Abracadabra." Nowadays, this same word is given a different connotation. It is associated with nonsense. However, whether this word possessed some supernatural powers, or it was a means for earning a livelihood for some medical quacks, is immaterial. The mathematical properties of an amulet which was based on this word are very interesting and instructive. Generally, this amulet was triangular in form as shown below.

A B R A C A D A B R A  
A B R A C A D A B R  
A B R A C A D A B  
A B R A C A D A  
A B R A C A D  
A B R A C A  
A B R A C  
A B R A  
A B R  
A B  
A

It will be noted that the word "Abracadabra" can be read in several different ways. The triangle contains 11 rows, each of them starting with the letter "A." Thus, if we start with this letter, we can read the word in a certain number of ways. We may start with two letters, with three letters, and so on. The total number of ways the word "Abracadabra" can be read is  $2^{10} = 1024$ .

Suppose that we have the word "Teacher." The "magic triangle" based on this word is

T E A C H E R  
T E A C H E  
T E A C H  
T E A C  
T E A  
T E  
T

The number of ways the word "Teacher" can be read is  $2^6 = 64$ .

Generally, the triangular writing of a word with  $n$  letters permits  $2^{n-1}$  different ways of reading that word. Thus, the greeting "Merry Xmas" may be written as

M E R R Y X M A S  
M E R R Y X M A  
M E R R Y X M  
M E R R Y X  
M E R R Y  
M E R R  
M E R  
M E  
M

and this message can be read in  $2^8 = 256$  different ways.

The greeting "Merry Xmas" may be also written in the rectangular form

M E R R Y X  
 E R R Y X M  
 R R Y X M A  
 R Y X M A S.

The number of ways this message may be read in this case may be determined if we consider a simple analysis. Suppose that we have the word "Merry" and we write it as follows

M E R R  
 E R R Y.

This rectangular arrangement permits only four ways of reading the word "Merry." The word "Teacher" may be written in rectangular form as

T E A C H  
 E A C H E  
 A C H E R.

Starting with the letter "T" in the first row, the word can be read in 5 different ways. Starting with two letters "TE" in the first row, the word can be read in 4 different ways. Starting with three letters "TEA" in the first row the word can be read in 3 different ways. This can be continued with four letters in the first row, and finally with five letters in the first row. Thus, the total number of ways the word "Teacher" can be read is

$$5+4+3+2+1=15.$$

Generally, for any three-rowed rectangle in which there are  $n$  letters in a row, the total number of ways of reading is

$$n+(n-1)+(n-2)+\cdots+2+1 = \frac{1}{2}n(n-1).$$

If we have a four-rowed rectangle with  $n$  letters in each row, then the total number of ways of reading is

$$\begin{aligned} & \frac{1}{2}[(n+1)n+n(n-1)+(n-1)(n-2) \\ & + (n-2)(n-3)+\cdots+3\cdot 2+2\cdot 1] \\ & = \frac{1}{6}n(n+1)(n+2). \end{aligned}$$

The total number of possible readings for a five-rowed rectangle with  $n$  letters in each row is

$$\frac{1}{24}n(n+1)(n+2)(n+3).$$

Note that the coefficients of these formulas have denominators whose factors are:

$$(1\cdot 2), (1\cdot 2\cdot 3), (1\cdot 2\cdot 3\cdot 4).$$

Thus, if we generalize, we may write the formula for the total number of possible reading for an  $m$ -rowed rectangle with  $n$  letters in each row

$$\frac{n(n+1)(n+2)(n+3)\cdots(n+m-2)}{1\cdot 2\cdot 3\cdot 4\cdots(m-1)}.$$

Multiplying the numerator and denominator by

$$1\cdot 2\cdot 3\cdot 4\cdots(n-1)$$

we obtain another form for the same formula

$$\frac{1\cdot 2\cdot 3\cdot 4\cdots(n+m-2)}{[1\cdot 2\cdot 3\cdot 4\cdots(n-1)]\cdot[1\cdot 2\cdot 3\cdot 4\cdots(m-1)]}$$

which may be written in factorial form as

$$\frac{(n+m-1)!}{n!\cdot m!}.$$

Thus the message "Merry Christmas" may be written as

M E R R Y C H R  
 E R R Y C H R I  
 R R Y C H R I S  
 R Y C H R I S T  
 Y C H R I S T M  
 C H R I S T M A  
 H R I S T M A S  
 R I S T M A S !

and it can be read in

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14}{(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)} = 17,160 \text{ different ways.}$$

Problems of this type are suitable for recreational material in intermediate algebra in connection with the topic on arithmetic progressions.

## MATHEMATICAL MISCELLANEA

Edited by PHILLIP S. JONES  
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### 62. A Method of Constructing a Triangle When the Three Medians are Given

College geometry texts commonly give two methods for constructing a triangle when the three medians are given. One method makes use of the whole median to fix an auxiliary triangle; the other uses two-thirds of each median for the auxiliary triangle. Both of these methods are given in Nathan Altshiller-Court's *College Geometry*, section 76, page 79.

The method which follows makes use of one-third of each median for the auxiliary triangle, and is the method by which I first solved the problem. Later I found the other two constructions given in texts, but have not seen this one in print. With a little planned introductory work the better high school students can be led to discover this method. In my classes in college geometry when the members of the class have mastered the two methods in the Altshiller-Court text, I suggest that they try the construction using one-third of each median. This requires their making an original analysis of the problem.

If a choice of method is given on a test, almost without exception my students use thirds of the medians. From this I conclude it must be the easiest, so I submit it in the hope that the simple construction may prove useful to other geometry teachers in high school or college.

**ANALYSIS:** Let us suppose we have the problem solved. We, therefore, have the completed triangle  $ABC$  (Figure 1) with the medians  $AA'$ ,  $BB'$ , and  $CC'$  meeting at  $G$  so that  $GA':GA:AA'=1:2:3$ ,  $GB':GB:BB'=1:2:3$ , and  $GC':GC:CC'=1:2:3$ . Suppose  $D$  is taken as the midpoint of  $GC$ . Draw  $A'D$ . Consider triangle  $BGC$ .

$$A'D = \frac{1}{2}BG, \quad BG = \frac{2}{3}BB', \quad A'D = \frac{1}{3}BB'$$

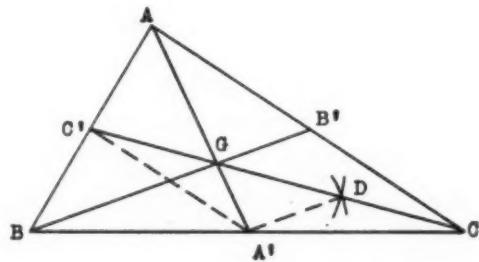


FIG. 1

$GD = \frac{1}{2}GC = \frac{1}{3}CC'$ ,  $GA' = \frac{1}{3}AA'$ . Hence, triangle  $GA'D$  is fixed since its sides are equal respectively to  $\frac{1}{3}AA'$ ,  $\frac{1}{3}BB'$  and  $\frac{1}{3}CC'$ . Consequently, the points  $C$  and  $C'$ , and  $A$  and  $A'$  are determined.  $AC'$  and  $CA'$  meet at  $B$ , hence  $ABC$  is determined.

**CONSTRUCTION:** Given the three medians  $m_a$ ,  $m_b$ ,  $m_c$ , construct triangle  $ABC$ . Divide each of the medians into three equal parts, by the usual construction. Draw  $AA' = m_a$ . From  $A'$  locate  $G$  so that  $A'G = \frac{1}{3}m_a$ . With  $G$  as center draw an arc of a circle with radius equal to  $\frac{1}{3}m_c$ ; with  $A'$  as center draw arc of circle with radius equal to  $\frac{1}{3}m_b$ . Let the two arcs intersect at  $D$ . Draw  $GD$  and extend through  $G$  and through  $D$  so that  $C'G = CD = GD$ . Draw  $AC'$  and  $CA'$  until they intersect at  $B$ . Draw  $AC$ .  $ABC$  is the desired triangle.

**PROOF:** Consider triangles  $GC'A'$  and  $GCA$ .

$$\frac{GC'}{GC} = \frac{GA'}{GA} = \frac{1}{2} \text{ by construction,}$$

$\angle A'GC' = \angle AGC$  since vertical angles are equal. Hence  $\triangle GC'A' \sim \triangle GCA$ .

Hence,  $A'C'/AC = \frac{1}{2}$ , and  $\angle GAB' = \angle GAC'$ . Thus  $A'C' \parallel AC$ , and  $\triangle BC'A' \sim \triangle BCA$ . This implies that

$$\frac{BA'}{BD} = \frac{BC'}{BA} = \frac{A'C'}{AC} = \frac{1}{2}$$

and it follows that  $A'$  and  $C'$  are the midpoints of  $BC$  and  $BA$  respectively.  $AA'$  and  $CC'$  are then medians of  $\triangle ABC$ .

The line  $BGB'$  is then the median upon  $AC$  since the medians of a triangle are concurrent.

Consider triangle  $BGC$ .  $D$  and  $A'$  are the midpoints of  $GC$  and  $BC$  by construction and by definition of a median respectively, and  $BG = 2A'D$ .

But,  $A'D = \frac{1}{3}m_b$  by construction, and thus  $BG = \frac{2}{3}m_b = \frac{2}{3}BB'$ , or  $BB' = m_b$ . Also  $CC' = m_c$ , and  $AA' = m_a$  by construction.

Therefore the medians of triangle  $ABC$  equal respectively the given medians  $m_a$ ,  $m_b$ ,  $m_c$  and our construction is proved.

The construction is possible when the sum of any two medians is greater than the third.

**SUGGESTIONS:** If the students have acquired facility in trisecting a line, it saves time to give one-third of each median instead of the whole medians. A line can be extended to three times its length more quickly than it can be trisected.

I find it very helpful to use different colors to indicate the given parts in construction problems. The parts used in construction are colored in the same way on the completed work. The work can thus be followed more readily and less oral explanation is required. When students color their construction it facilitates checking their work, since one can see at a glance the method employed. The use of color also adds to the attractiveness of the paper. The best papers are posted on the bulletin-board. This serves as a stimulus to others to strive for neatness and accuracy.

Colored chalk is a *must* on my list of class room supplies. Each student is required to have a box of colored pencils. Besides using color for construction work I use it in demonstration to mark corresponding parts in similar and congruent figures. The pupils find it useful in analyzing originals in these units.

SISTER MARY CONSTANTIA, S. C. L.  
The Saint Mary College  
Xavier (Leavenworth Co.), Kansas

### 63. A Further Note on Nedians

In a communication to **THE MATHEMATICS TEACHER** in the issue of January 1951 I drew attention to some of the properties of the nedians<sup>1</sup> of a plane triangle.

The nedians may be defined as the straight lines which are drawn from the vertices  $A$ ,  $B$ ,  $C$  of a  $\triangle ABC$  (Fig. 2) to

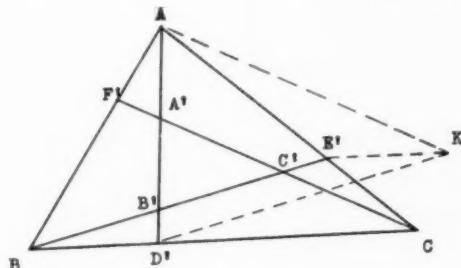


FIG. 2

points  $D'$ ,  $E'$ ,  $F'$  in the sides  $BC$ ,  $CA$ ,  $AB$  respectively which divide these sides in the ratio

$$\frac{BD'}{BC} = \frac{CE'}{CA} = \frac{AF'}{AB} = \frac{1}{N}.$$

The lengths of these nedians are denoted by  $n_1$ ,  $n_2$ ,  $n_3$  and it is easy to show by the use of Stewart's Theorem that

$$(1) \quad n_1^2 + n_2^2 + n_3^2 = \frac{N^2 - N + 1}{N^2} (a^2 + b^2 + c^2).$$

The medians  $AD$ ,  $BE$ ,  $CF$  drawn from  $A$ ,  $B$ ,  $C$  to the midpoints  $D$ ,  $E$ ,  $F$  of  $BC$ ,  $CA$ ,  $AB$  respectively (Fig. 3.) are a special case of the nedians and are obtained by putting  $N = 2$  in the formula above. Hence, if  $m_1$ ,  $m_2$ ,  $m_3$  are the lengths of the medians:

$$(2) \quad m_1^2 + m_2^2 + m_3^2 = \frac{3}{4} (a^2 + b^2 + c^2),$$

a well-known expression.

The intersecting nedians form a tri-

<sup>1</sup> I coined the word *nedians*. Hence I prefer it to the alternatives.

*Editor's note:* Professor Satterly touched off a series of notes on *nedians*, *redians*, *cevians* with a note in **THE MATHEMATICS TEACHER**, XLIV (Jan., 1951), p. 46 ff. Further discussion appeared in the same volume (May 1951), p. 310 ff.; (Nov. 1951), p. 496 ff.; (Dec. 1951), p. 559 ff., and in volume XLV (Jan. 1952), p. 44 ff.

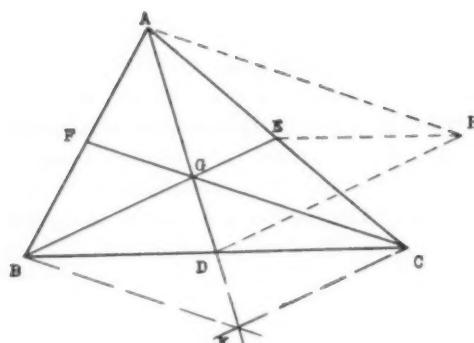


FIG. 3

angle  $A'B'C'$  which I called the *Nedian triangle*. Its area may be obtained by using coordinate geometry having first found the equations of the nedians. The relation is:

$$(3) \quad \begin{aligned} & \text{Area of the median triangle} \\ & = \frac{(N-2)^2}{N^2-N+1} \cdot (\text{area of the triangle } ABC). \end{aligned}$$

Also once the coordinates of the vertices of the median triangle have been found it is easy to show that the centroid of the median triangle is coincident with the centroid of the triangle  $ABC$ . These last two theorems may be found in some textbooks and in such collections as Alliston's *Mathematical Snack Bar*.<sup>2</sup>

Lately my attention was called to the triangle formed by taking the medians as sides. An example in Hobson's *Trigonometry* stated that this area, which may, of course, be expressed as

$$(4) \quad \begin{aligned} & [\sigma(\sigma-m_1)(\sigma-m_2)(\sigma-m_3)]^{1/2} \\ & = \frac{3}{4}(\text{Area of } \triangle ABC) \end{aligned}$$

where  $2\sigma = m_1 + m_2 + m_3$ . The factor  $\frac{3}{4}$  is equal to  $(N^2-N+1)/N^2$  when  $N=2$ , and starting from this I wondered whether the triangle formed by taking the nedians as sides which of course, is expressed by

$$(5) \quad [\sigma_1(\sigma_1-n_1)(\sigma_1-n_2)(\sigma_1-n_3)]^{1/2}$$

where  $2\sigma_1 = n_1 + n_2 + n_3$  could be expressed as

<sup>2</sup> Norman Alliston, *A Mathematical Snack Bar*. (W. Heffer and Sons Ltd., 1936) p. 11 ff.

$$(6) \quad \frac{N^2-N+1}{N^2} \times (\text{Area of the } \triangle ABC).$$

I found this to be true and it constitutes a very pretty theorem.

During this work I found and used diagrams which I had not seen before. It is well known that the medians divide the  $\triangle ABC$  into six equal areas, also that if (Fig. 3)  $BK, CK$  are drawn parallel to  $CF, BE$  respectively, they meet on  $AGD$  produced and

$$(7) \quad \text{Area of } \triangle GKC = \frac{1}{3}(\text{Area of } \triangle ABC).$$

Now  $GK = (2/3)m_1, KC = (2/3)m_2, CG = (2/3)m_3$ . Thus

$$(8) \quad \begin{aligned} & (\text{Area of } \triangle GKC) = \left[ \frac{2\sigma}{3} \left( \frac{2\sigma}{3} - \frac{2m_1}{3} \right) \right. \\ & \left. \left( \frac{2\sigma}{3} - \frac{2m_2}{3} \right) \left( \frac{2\sigma}{3} - \frac{2m_3}{3} \right) \right]^{1/2}. \end{aligned}$$

Formula (4) now follows from (7) and (8).

Another way of getting the same result is to note that if we take the triangle  $ABC$  with its medians (Fig. 3) and draw  $EH$  parallel to  $BC$  and make it of length  $BD$  and then join  $HA$ , we find  $HA$  to be equal and parallel to  $CF$ . Thus  $\triangle ADH$  is the triangle formed by taking the medians as sides. Using coordinate geometry we get the coordinates of  $A, D$ , and  $H$ , and the area comes out as above by the usual formula.

The same method may be used with the nedians as shown in Figure 2. Draw  $D'K \parallel BE'$  to meet  $E'K$  drawn  $\parallel BC$  at  $K$ . Join  $KA$ . The sides of the triangle  $AD'K$  are the nedians  $n_1, n_2, n_3$ , and having gotten the coordinates of the vertices  $A, D', K$  the area of the triangle  $AD'K$  is soon found and we have the new theorem: *If  $n_1, n_2, n_3$  are the nedians of a  $\triangle ABC$  the area of the triangle formed by taking the nedians as sides is  $(N^2-N+1)/N^2$ . (Area of the  $\triangle ABC$ .)*

Perhaps our readers will tell us if my theorem is new. One often "discovers" new theorems and then finds them tucked

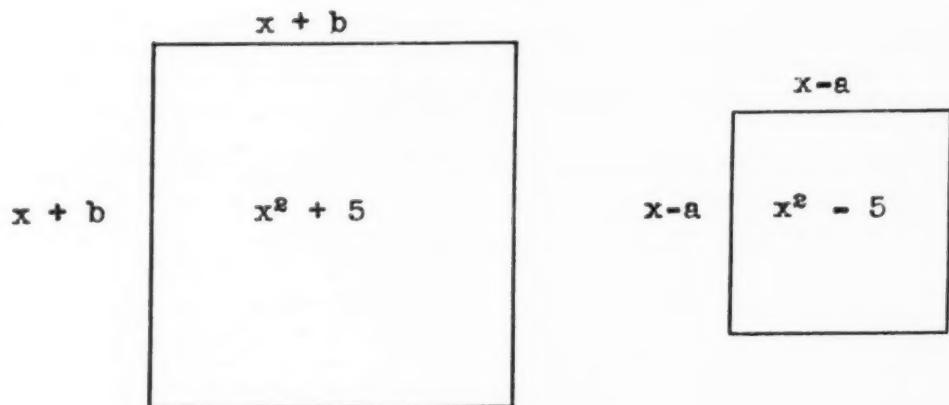


FIG. 4

away in some textbook of ancient vintage. and

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#### 64. A Historic Problem

A scientific tournament was staged on the occasion of the presentation of the famous mathematician LEONARDO OF PISA or FIBONACCI (c. 1170-c. 1250) to Emperor Frederick II of Hohenstaufen. John of Palermo, an imperial notary, proposed several problems to Leonardo. One of these was to find a number,  $x$ , such that  $x^2 + 5$  and  $x^2 - 5$  are both perfect squares.<sup>3</sup>

Leonardo worked the problem by building squares by the summation of odd numbers, a method original with him although the problem had been solved earlier by Arabic mathematicians. Several years ago, I hit upon the following solution by employing a Diophantine process.

Let  $x$  = the number

$x+b$  = side of the square whose area is  $x^2+5$  (Fig. 4), and

$x-a$  = side of the square whose area is  $x^2-5$ .

then:

$$(1) \quad \begin{aligned} (x+b)^2 &= x^2+5 \\ x^2+2bx+b^2 &= x^2+5 \\ x = \frac{5-b^2}{2b} \end{aligned}$$

<sup>3</sup> Florian Cajori, *A History of Mathematics*. (New York: The Macmillan Company, 1938), p. 124.

$$(2) \quad \begin{aligned} (x-a)^2 &= x^2-5 \\ x^2-2ax+a^2 &= x^2-5 \\ x = \frac{5+a^2}{2a} \end{aligned}$$

Equating (1) and (2), and solving for  $b$ , we have

$$(3) \quad \begin{aligned} \frac{5-b^2}{2b} &= \frac{5+a^2}{2a} \\ 5a-ab^2 &= 5b+a^2b \\ ab^2+b(5+a^2) &= 5a \\ b^2+\frac{b}{a}(5+a^2) &= 5 \\ b^2+\frac{b}{a}(5+a^2)+\frac{1}{4a^2} &= \frac{1}{4a^2}(5+a^2)^2 \\ &= 5+\frac{1}{4a^2}(5+a^2)^2 \\ b+\frac{1}{2a}(5+a^2) &= \pm \frac{a^4+30a^2+25}{4a^2} \\ b = -\frac{5+a^2}{2a} \pm \frac{1}{2a}\sqrt{a^4+30a^2+25} & \end{aligned}$$

The  $b$  we seek is a positive rational number.  $b$  will be rational if we can find a value for  $a$  which will clear the radical. By trial or solving, we find  $a$  to be 6 or 0. We reject 0 since it would be used as a divisor in solving for  $b$ . Using 6, the radical clears and becomes 49. Substituting 6 in

equation (2), we have

$$(4) \quad x = \frac{5+a^2}{2a} = \frac{5+36}{2} = \frac{41}{12} \text{ Q.E.D.}$$

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*65. The Treatment of Some Common Difficulties*

THE "UNDERSTOOD" 1: In arithmetic emphasis upon the fact that any integer is equivalent to a fraction with the integer in the numerator and 1 in the denominator will help substitute understanding for standard difficulties. Likewise in algebra many students fail to realize that the exponent in quantities such as  $x$ ,  $y$ , 5, etc., is understood to be 1. Actually writing the 1 in cases involving exponents and logarithms is helpful. Thus, the student can see that  $x^1 \cdot x^3 = x^4$ , just as he can understand that  $x^2 \cdot x^5 = x^7$ . In calculus, the differentiation of an expression such as  $4x$  is made clearer by thinking of  $4x^1$ , and then applying the general rule:  $1 \cdot 4x^{1-1}$ , or finally  $4x^0$  or 4. It should also be emphasized that 1 is "understood" as a coefficient in such expressions as  $x$ ,  $y^2$ , 2. Actually writing the 1 a few times in the early stages of teaching operations with literal numbers is a helpful device.

ZERO EXPONENTS AND COEFFICIENTS: Actually writing zero exponents and coefficients is helpful in teaching long division with polynomials, and in showing series of ascending and descending powers. Thus, dividing  $a^3 - b^3$  by  $a - b$  may be made clearer by writing the dividend as  $a^3 - 0a^2b - 0ab^2 - b^3$ . We really have added nothing, but this device makes the work easier. Greater understanding is obtained by writing the dividend as  $a^3b^0 - 0a^2b^1 - 0a^1b^2 - a^0b^3$ , which shows the systematic and orderly changing of the exponents. After the division procedure is understood, the student may omit the writing of the terms amounting to 0, but he will then understand why spaces should be

allowed in the dividend for the missing terms.

THE SIGNIFICANCE OF THE NUMBER  $e$ : Students in calculus will find that drawing the graph of  $y = e^x$  will enable them to see that the derivative of this function is also  $e^x$ . Actual measurement and computation of the slope of tangents drawn to the curve at several points will furnish visible and numerical evidence that the derivative is actually  $e^x$  or the  $y$ -value of the function at the point of tangency. Drawing the graphs of  $y = e^{1/2x}$  and  $y = \sin 2x$  will likewise convince students that the derivatives of these functions are respectively  $1/2e^{1/2x}$  and  $2 \cos 2x$ . By constructing the graph of  $y = \log x$ , a student can see that the derivative of this function is  $(1/x)dx$ , and conversely, that the integral of  $dx/x$  is  $\log x$  plus a constant. In fact, graphical and numerical procedures of this sort not only add to the understanding of the particular topic, but also re-emphasize some of the basic principles inherent in all differentiation and integration procedures as well as suggesting the graphical and numerical procedures which, in more elaborate form, are so important in applied mathematics.

The fact that the derivative of an exponential function is proportional to the function itself is a restatement of the first sentence above which should be formulated with students and then written as the differential equation  $y' = ky$  which in turn should be associated with several of its many important physical interpretations such as the variation of atmospheric pressure with altitude (or pressure on the ears with the depth at which one swims in the pool), growth, and decay.

From the graph, however, much of this function's behavior and its connection with physical situations may be pointed out to and understood by secondary school as well as college students without the terminology and notation of the calculus.

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## APPLICATIONS

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### Ar. 20 Gr. 8-12 A Water Main Breaks Under Test Pressure

The following item is quoted by permission of the City Editor from the August 16 issue of the *Columbus Dispatch*.

City workman resumed pressure testing of the new \$1,000,000 North Side force main Friday after replacement of an 18-foot faulty section which blasted open Thursday under 150 pounds per square inch pressure.

Shop Superintendent John Cook of the Division of Water said the job was completed within six hours after the blast, which threw water and chunks of dirt some 40 feet. It ripped open a section of the 20-inch,  $\frac{1}{4}$  inch thick main just south of W. North Broadway along the west side of the Olentangy River Road.

The explosion occurred at 2 P.M. as engineers were running a pressure test from Lane Avenue at the River Road. Leo Dallas, Job foreman, said the line held the 150-pound pressure for about 15 minutes.

"Then she let go," Dallas said. "We knew it right away. It showed on the gauges."

After reading the above item, I began wondering what the total force had been on the inner surface of the 18-foot section, with 20-inch internal diameter, and 150 pounds per square inch. The problem makes an ideal application of arithmetic for students in grades 8 through 12.

The problem involves finding the number of square inches of lateral area of a cylinder and then multiplication by the number of pounds on each square inch. Following are the calculations.

$$\text{Lateral area} = 20\pi \times 18 \times 12$$

$$\text{Total force in pounds} = 20\pi \times 18 \times 12 \times 150$$

$$\text{Total force in pounds} = 2,030,000$$

In tons this comes to over one thousand. An amazing thing about the whole problem is that one man could impose this gigantic force on the main, providing the main was completely filled with water and

a quarter or half square inch plunger used. This is because of a well known principal of hydraulics, Pascal's Law, which states that pressure applied to any portion of a confined liquid is transmitted undiminished to all portions of the liquid.

### P.G. 11 Gr. 10-11 Cake Icing

KENNETH SWALLOW, who a short time ago privately published a geometry workbook containing flexible cardboard devices for the experimental discovery of theorems, suggested the following interesting problem concerning the cutting of cake in the Ohio Union cafeteria. It seems that cake in the form of a rectangular solid with a square perimeter is heavily iced and served in slices as one of a number of desserts in the Union cafeteria. The slicers, being oblivious to the exigencies of Euclid, usually employ parallel cuts from one end to the other, thus creating two slices with an inordinately large amount of icing on their sides. This causes hungry cake-minded students to vie for the coveted pieces.

The problem posed by Mr. Swallow is simply this: How may the cake be easily cut, assuming no thickness to the icing, so that each piece will have an equal amount of both cake and icing?

I plan to diagram and explain the solution in the February, 1953 department.

### Ar. 21 Gr. 7-12 Jet-Age Mathematical Terms

With the advent of the air-age and more recently the jet-age it behooves mathematics teachers to keep abreast of new terminology which can best be defined mathematically. Following is a list of

some of these terms. With our American children already diligently exploring outer space well ahead of the armed forces, the clarification of these terms in school is timely and self-motivated.

G's						
Speed in Mach numbers or ratios						
Surface friction						
Trajectory						
Collision course						
Pressurized cabin						
Lift						
Thrust						
Drag						
B.t.u., Calorie						
Wing load						
Orbit						
Engine efficiency						
Momentum						
Dihedral angle of wings						
Sweep back						
Triangle design						
Aspect ratio						
Angle of attack						
Radius of action						
Center of gravity, Centroid						
Turning radius						
Sidereal time						
Light year						
Angstrom unit						
Elliptical						
Rate of climb						
Inertia						

#### Ar. 22 Gr. 6-9 *Fractions and Basketball Scores*

A person interested in basketball once said that the difference between the points earned by a field goal and a foul shot ought not to be so great. He proposed that each field goal should count  $1\frac{1}{2}$  points and each foul shot should count  $1\frac{1}{2}$  points. On January 10, 1949, Ohio State lost a close game with Illinois. Below is the box score of this game. What would be the individual and total scores of the game with the above method of scoring? Would this change the final outcome? Do you think the new method of scoring is better?

OHIO STATE	FG	F	MF	PF	TP	TP (New System)
Donham, f	4	5	5	3	13	
Schnittker, f	8	4	6	5	20	
Jacobs, f	0	0	0	0	0	
Raidiger, c	7	3	1	4	17	
Giacomelli, c	0	0	1	2	0	
Brown, g	2	1	0	4	5	
Burkholder, g	3	2	2	4	8	
Totals	24	15	15	22	63	

ILLINOIS	FG	F	MF	PF	TP	TP (New System)
Eddleman, f	4	2	4	5	10	
Foley, f	0	0	0	0	0	
Marks, f	1	0	2	2	2	
Anderson, f	1	0	0	1	2	
Osterkorn, c	7	4	3	4	18	
Green, c	5	2	0	3	12	
Sunderlage, g	1	2	0	4	4	
Thurlby, g	2	0	0	4	4	
Erickson, g	4	4	4	3	12	
Totals	25	14	13	26	64	

The abbreviations above each column are as follows: FG—Field Goals, F—Foul Shots Made, MF—Missed Fouls, PF—Personal Fouls, TP—Total Points.

You are to fill in the column marked TP (New System).

#### Ar. 23 Gr. 5-8 "Sleepers"

Too often pupils begin to operate at the mere sight of data without thinking about the problem. I have successfully alerted pupils to the need for preliminary reflection by inserting in a group of orthodox problems one of the following so-called "sleepers." While I do not believe in ridicule, the humor of missing a "sleeper" serves to make the point subtly and harmlessly.

- If one broad jumper can jump a ditch 6 feet wide, how wide a ditch can seven broad jumpers jump?
- If the speed limit is 30 miles per hour on route 20, what is the speed limit on route 40?
- If Henry I had 3 wives, how many wives did Henry VIII have?
- If a broad jumper can jump 14 feet after running 30 feet, how far could he jump after running 60 feet?
- A rooster weighs  $5\frac{1}{4}$  pounds while standing with both feet on a scale platform. How much would the rooster weigh if he stood with only one foot on the scale platform and the other tucked under his wing?

#### Al. 17 Gr. 11-12 *Use of Proportions in Chemistry*

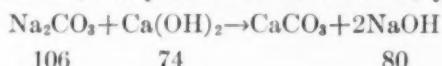
One of the most important applications of proportions occurs in chemistry when the weight of one reacting substance is calculated from the known weights of other substances in the reaction. These calculations are known as "stoichiometric calculations" and provide the basis of

quantitative analysis. Following is a typical problem:

The equipment in the lime-soda ash method of making caustic soda in a certain factory can handle only one hundred pounds of raw material at a time. How many pounds of lime and how many pounds of soda ash must be used, and how many pounds of caustic soda (lye) are produced?

The first task is to write and balance the equation for the reaction:

soda ash + lime yields calcium carbonate + lye



The formula weights are written under each ingredient above. These formula weights are equal to the molecular weights of each compound times the number in front of the compound needed to balance the equation. The molecular weights are found by adding the atomic weights within each compound. For example, the formula weight, 80, for lye is found by adding the atomic weights of sodium, oxygen, and hydrogen:  $23 + 16 + 1 = 40$ . The molecular weight of lye is 40. The formula weight is obtained by doubling this, since two molecules are involved in the balanced formula.

The solution of the problem hinges on the chemical principle that the weights of the reacting substances have the same ratio as the formula weights of these substances. Following is the algebraic solution:

Let  $x$  = weight in pounds of soda ash.

Then  $100 - x$  = weight in pounds of the lime.

By the above principle:

$$\frac{x}{100-x} = \frac{106}{74}$$

$x = 59$  pounds of soda ash

$100 - x = 41$  pounds of lime.

Another proportion determines the amount of lye produced, assuming one hundred per cent yield.

Let  $x$  = amount of lye produced in pounds

$$\frac{x}{59} = \frac{80}{106}$$

$x = 45$  pounds of lye.

#### Al. 18 Gr. 9-12 School Problems in Algebra

Let us face it. Most of our algebra verbal problems are deadening. Why not occasionally give some problems of the following type?

1. Gay had 5 minutes available between class bells. She used the time as follows:  $s$  seconds to leave the classroom; twice as many seconds to greet a friend; 4 times as much to make a date with a ?;  $s$  seconds to walk to her locker; a trip to the central office and back to the door of her next class required twice as much time as the time with the ?; she walked into the classroom and sat down in  $4s$  seconds ready for instruction. How many seconds did she use for each part of her busy 5 minutes?

2. Jerry went to a 60 minute study hall with his science and English books. He managed to devote  $\frac{1}{2}$  as much time to the study of English as to the study of science. How much time did he devote to each?

3. Phil wanted to build a bookcase. He asked for an estimate of the items for building it. His approximate information ran as follows:

lumber—the biggest item of expense  
nails—about  $1/20$  of the lumber  
glue—about  $1/40$  of the lumber  
paint—about  $1/10$  of the lumber

If the bookcase would cost about \$9.40 about how much is the cost of each item?

4. On Monday, November 17, the school health office reported a minimum number of students with colds. The next day the number of colds had doubled. On November 19, the number was the same as on November 18. On November 20, the cases of colds had trebled over November 17. November 21 was the same as November 20. The total cases for the week were 99. How many colds were there on each day?

"Mathematics involves the development of the most precise, most sustained, most closely knit set of habits."

—Henry Link, *The Return to Religion*. Macmillan, 1937, p. 153.

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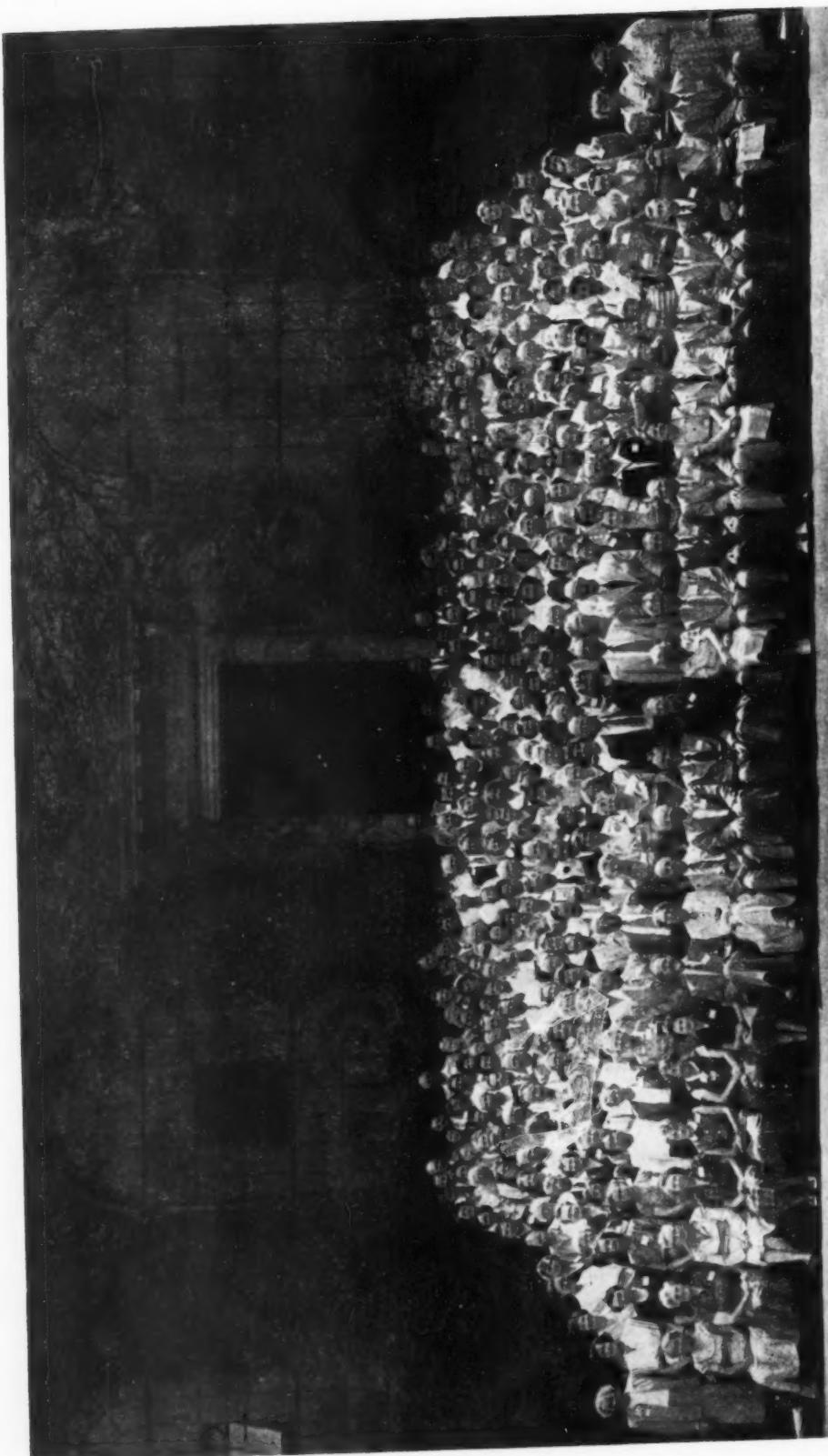
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(Continued on page 616)



1952 Joint Summer Meeting of the National Council of Teachers of Mathematics and the New England Institute for Teachers of Mathematics

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The Fifth Annual Institute for Teachers of Mathematics

of the

Association of Teachers of Mathematics in New England

will be held

August 20-27, 1953, Colby College, Waterville, Maine

## NEWS NOTES

The Joint Meeting of The National Council of Teachers of Mathematics (12th Summer Meeting) and The New England Institute for Teachers of Mathematics (4th Institute) sponsored by The Association of Teachers of Mathematics in New England was held at The Phillips Exeter Academy, Exeter, New Hampshire, August 21-28, 1952. The combining of the two meetings proved to be a great success. There was an enthusiastic group of about 500 present. The geographical area represented ranged from Ontario to the Canal Zone and from California to Maine.

The program opened on Thursday, August 21 with a banquet for The National Council with Professor Bancroft Brown, Dartmouth College, as speaker on the subject "Education in the Obvious." On each of the following days there were two lectures and three sets of study groups. Each set consisted of nine different groups discussing topics suitable for various school levels. After the National Council meetings were over on August 24, five groups in each set continued as the Institute groups and a new group particularly for elementary school teachers was added.

There were several rooms given over to the exhibition of materials used in the teaching of mathematics. These included displays of books by several publishing companies, commercial teaching aids, classroom furnishings, a collection of linkages, devices for use in the calculation of  $\pi$  by experiment, and many projects carried out by students in the form of posters, booklets, and models. An electrical calculating machine devised by two high school students drew a great deal of attention. There were many free pamphlets and other materials available for use in the classroom. Films and filmstrips were shown regularly.

There was ample opportunity for recreation.

A pastime room, full of mathematical devices and puzzles, was a popular gathering place between study groups. Following the afternoon meetings tea was served in The Big Room in Phillips Hall. A snack bar was open following the evening lectures, and many gathered there to continue friendly discussions.

Besides the daily opportunities for tennis, swimming, boating, or walking on the beautiful campus of the Academy, the group had a choice on Sunday of an all-day trip to the White Mountains or a boat trip to the Isles of Shoals. Evening entertainment included unusual magic tricks by Professor William F. Cheney, Jr., University of Connecticut, and colored slides of the 1951 Institute at Connecticut College for Women by Miriam Loring of the Belmont, Massachusetts High School.

On Tuesday, picnic lunches were provided, and many of the group visited the neighboring beaches, historic houses in nearby Portsmouth, Phillips Andover Academy, or the Towle Corporation.

The formal sessions of the Institute closed with a banquet on Wednesday evening, August 27, at which Dean John E. Burchard of the Massachusetts Institute of Technology was the speaker. It was a fitting conclusion to a week full of inspiration and contagious enthusiasm as well as congeniality and fellowship due in a large part to the hospitality offered by The Phillips Exeter Academy and its Principal, William G. Saltostall.

The members of the General Committee which planned the Joint Meeting were Jackson B. Adkins, General Chairman and Program Committee; Barbara B. Betts, Exhibits; Janet Height, Recreation; Dr. H. Gray Funkhouser, Housing; Christina S. Little, Laboratories; Henry W. Syer, Representative for The National Council; and Ruth B. Eddy, Publicity.

### Officers of Affiliated Groups

(Continued from page 613)

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MICHIGAN: Duncan A. S. Pirie, 4628 Devonshire Road, Detroit 24  
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MISSOURI: Miss Nellie Kitchens, 404 Frederick Apartments, Columbia  
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RHODE ISLAND: M. L. Herman, Moses Brown School, Providence  
SOUTH CAROLINA: Miss Lucile Huggin, R.F.D. 2, Spartanburg

(Continued on page 619)

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## BOOK SECTION

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Edited by JOSEPH STIPANOWICH  
Western Illinois State College, Macomb, Illinois

### BOOKS RECEIVED

#### High School

*General Mathematics at Work*, by Claude H. Ewing and Walter W. Hart. Cloth, v+266 pages, 1952. D. C. Heath and Co., 285 Columbus Ave., Boston 16, Mass. \$2.80.

#### College

*Calculus*, by Tomlinson Fort, University of Georgia. Cloth, xii+560 pages, 1951. D. C. Heath and Co., 285 Columbus Ave., Boston 16, Mass. \$4.75.

*College Geometry* (Second ed. rev.) by Nathan Altshiller-Court, University of Oklahoma. Cloth, xix+313 pages, 1952. Barnes and Noble, Inc., 105 Fifth Ave., New York 3, N. Y. \$4.00.

*General College Mathematics*, by W. L. Ayres, Cleota G. Fry, and H. F. S. Jonah, all of Purdue University. Cloth, xii+283 pages, 1952. McGraw-Hill Book Co., 330 W. 42nd St., New York 36, N. Y. \$3.75.

*Modern Elementary Statistics*, by John E. Freund, Alfred University. Cloth, x+418 pages, 1952. Prentice-Hall, Inc., 70 Fifth Ave., New York 11, N. Y. \$5.50.

*Basic Mathematics for Engineering and Science*, by Walter R. Van Voorhis, and Elmer E. Haskins, both of Fenn College. Cloth, x+619 pages, 1952. Prentice-Hall, Inc., 70 Fifth Ave., New York 11, N. Y. \$5.75.

*Methods of Applied Mathematics*, by F. B. Hildebrand, Massachusetts Institute of Technology. Cloth, xi+523 pages, 1952. Prentice-Hall, Inc., 70 Fifth Ave., New York 11, N. Y. \$7.75.

*Theory of Numbers*, by B. M. Stewart, Michigan State College. Cloth, xiii+261 pages, 1952. Macmillan Co., 60 Fifth Ave., New York 11, N. Y. \$5.50.

#### Miscellaneous

*Articles on the History of Mathematics: A Bibliography of Articles Appearing in Five Periodicals*, by Cecil B. Read, University of Wichita. University Studies No. 26, paper, 32 pages, 1952. The Municipal University of Wichita, Wichita, Kansas. Single copy, \$0.50; additional copies, \$0.25.

*Mathematical Models*, by H. Martyn Cundy and A. P. Rollett. Cloth, 240 pages, 1952. Oxford University Press, 114 Fifth Ave., New York 11, N. Y. \$5.50.

### REVIEWS

*Growth in Arithmetic* (Grade 7), John R. Clark, Rolland R. Smith, and Harold E. Moser. Yonkers-on-Hudson, New York, World Book Company, 1952. v+314 pp., \$2.12.

The book is very well adjusted to the developmental level of seventh-grade students. An extremely clear and meaningful review is presented in the first few chapters. There is an abundance of drill exercises and many problems relating to the experience of an adolescent child. It would seem that this book, because of the number and diversity of problems, could be adjusted to suit the needs of varying levels of arithmetic abilities.

Maintenance of skills and understandings is provided for by well-distributed practice exercises. The units of work are short enough to cope with the attention span of boys and girls of adolescent age.

The format would be especially attractive to students as the book is nicely illustrated and unusually legible. The publisher employs the two column system of printing so that the book lies flat when opened. The colorful illustrations will probably be helpful in motivating the subject.

Teachers will find this book very helpful in teaching a course in seventh grade arithmetic that will meet the needs of a wide range of students.—WILLIAM H. NAULT, W. K. Kellogg Junior High School, Battle Creek, Michigan.

*Second Algebra* (2nd. ed.), Virgil S. Mallory and Kenneth C. Skeen. Chicago, Benj. H. Sanborn & Co., 1952. vii+480 pp., \$2.48.

The Mallory-Skeen Second Algebra will be welcomed by teachers who have found the Stone and Mallory and later the Mallory algebras successful teaching texts, and will undoubtedly add new friends.

Features of the book are inventory tests labeled "Can You Do These?" for reviewing first-year algebra and refreshing in concepts and skills needed in approaching a new topic, boxed examples for making processes clear, and the use of simple drawings and suggestions for classifying the data in solving world-problems. Practice exercises are graded to three levels of difficulty, and there are additional optional topics. Provision is made for reviews and main-

tenance, with a program of cumulative reviews and tests, including a "Keeping Up in Arithmetic" test, at the end of each chapter.

Chapters 1-6 are largely a review of first year algebra from a more advanced point of view. Chapters 7-11 develop intermediate algebra. Chapter 12 contains supplementary review exercises and optional topics including the factor and remainder theorems and synthetic division, determinants and their use in solving systems of equations, differentiation of polynomials and maximum and minimum problems, and integration of simple forms and finding the area under a curve.

Points of improvement over the previous Mallory texts include distinguishing between sequence and series, stating the commutative, associative, and distributive laws, use of the word "complex" rather than "imaginary" in connection with negative discriminant of a quadratic, more emphasis on the topic of variation with added practice exercises, and a more interesting introductory discussion of conic sections.

The authors deserve special commendation in the matter of the readability of the text. Recognizing that the usual mathematical difficulties are often complicated by reading difficulties, they have given special attention to semantics, vocabulary, and sentence-structure, with the result that the book satisfies established criteria for readability.

The format of the book is attractive.—  
KATHARINE E. O'BRIEN, Deering High School,  
Portland, Maine.

*How Big? How Many? Arithmetic for Home and School*, Gladys Risden. Boston, The Christopher Publishing House, 1951. 248 pp., \$3.50.

This book may be considered as a guide book

for those individuals who are directing children in the learning of arithmetic. It shows how children learn through experience by including numerous illustrations in which number facts and number relationships have been discovered. However, experience that ends with finding the answer is not sufficient. The child needs organized experience in taking groups apart and putting them together, both perceptual and non-perceptual, and guidance in generalizing upon this experience to build up a system of ideas, the system of ideas the race has evolved to serve in making understandable the quantitative situation of life.

Teachers of arithmetic will find this text contains many valuable suggestions for teaching the slow learner as well as those who are in need of remedial work. It should also be of great value to those who are in training to be teachers of arithmetic.—L. H. WHITCRAFT, Ball State Teachers College, Muncie, Indiana.

*Jacobian Elliptic Function Tables*, L. M. Milne-Thompson. New York, Dover Publications Inc., 1951. xi+123 pp., \$2.45.

This little book contains tables of the more frequently used elliptic functions and fills, at a moderate price, a need that has long been felt (especially among applied mathematicians). The tables themselves are easy to use. Approximately the first half of the book is devoted to a summary of the theory of elliptic functions and contains an extensive collection of useful formulas. Some of the applications to physical problems and conformal mapping are indicated. Although parts of the book are understandable to readers familiar with elementary calculus, nevertheless it is designed primarily for research workers in applied sciences, who should find it invaluable.—W. E. JENNER, Northwestern University, Evanston, Illinois.

## State Representatives

(Continued from page 617)

**SOUTH DAKOTA:** Miss Florence Krieger, Rapid City High School, Rapid City

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## RESEARCH IN MATHEMATICS EDUCATION

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*Edited by JOHN J. KINSELLA*

*School of Education, New York University, New York 3, N. Y.*

**The Question:** What content of general mathematics best serves the purposes of general education at the secondary education level?

**The Study:** Woodby, Lauren G. *A Synthesis and Evaluation of Subject-Matter Topics in Mathematics for General Education*. Ph.D. dissert. University of Michigan, Ann Arbor, 1952.

The purpose of this study was "to synthesize a comprehensive list of subject-matter topics in general mathematics from 10 contributing sources and to determine the relative values of these topics for general education in the secondary school." (Page 1).

General education was defined as "those phases of non-specialized, and non-vocational learning that should be the common experience of all educated men and women." (Page 53) "Educated" meant having "profited from the activities of the public school system (or its equivalent) through grade XIV." (Page 53) General mathematics was taken to be "non-compartmentalized, non-specialized, non-preparatory mathematics that is desirable for every person regardless of vocational and specialized needs." (Page 53) The secondary school was taken to include grades VII through XIV. In the study no attempt was made to determine grade or age level placement, order of presentation, or methods of instruction.

As a sub-problem to his main problem Dr. Woodby first sought to "compile a comprehensive list of subject-matter items from certain terminal courses in mathematics now being offered at the junior college level, and to evaluate these items as to their worth for general education."

(Page 56) Questionnaires received from 62 institutions ranging in enrollment from 125 to over 24,000, with a median enrollment of 944, revealed that only 5 of the 62 courses had been based on research investigations and that one or two textbooks were basic guides in all but 7 courses. From the 62 institutions 18 syllabi were available for analysis. The study of these was supplemented with an examination of 6 textbooks which had been used in 4 or more courses. The investigator emerged with 570 items classified under arithmetic (110), algebra (83), geometry (105), business and finance (62), trigonometry-analytic geometry-calculus (78), probability and statistics (41), history and biography (73) and miscellaneous (18). A jury of 11 specialists in mathematics and 8 in mathematics education rated the items for their value in a "mathematics for general education" one-year course at the junior college level. This reviewer noted that 138 items had a mean ranking closest to "essential" and 327 nearest to "of considerable value but not essential"; the remaining 105 items would have doubtful value for the course.

Realizing that this investigation would deal with only 2 of the 6 years of the secondary education period Dr. Woodby selected, by the use of criteria relevant to the purposes of his study, four research studies,<sup>1</sup> including his own, three committee reports<sup>2</sup> and three courses of study<sup>3</sup>

<sup>1</sup> Schorling, Raleigh. *A Tentative List of Objectives in the Teaching of Junior High School Mathematics*. Ann Arbor: George Wahr, 1925.

Richtmeyer, Cleon C. *Functional Mathematics Needs of Teachers*. Colorado State College of Education. 1937.

Moore, Vesper D. *The Mathematics of General Education for the Teacher*. Ph.D. University of Michigan, 1951.

to fill out his individual efforts. These "10 contributing sources" yielded 1077 topics which were ranked in importance by using the method followed by Curtis in a similar study in another field.<sup>4</sup> Next, 9 individuals who were considered experts in educational research and specialists in mathematics education were asked to give weights to the "10 contributing sources." From these data and an appropriate formula an adjusted weight for each of the 10 sources was determined. Then, the relative value of each topic under each source was obtained by multiplying its rank in the total of 1077 by the weight assigned to its source by the 9 authorities. When a topic appeared under more than one source a summation yielded its "aggregate value." Then, after arbitrarily assigning 100 to the topic having the highest aggregate value, all of the remaining topics were given "index values," expressed as integers less than 100, according to the ratio of their "aggregate values" to the highest aggregate value. Finally, the 1077 topics were organized under the categories of number and computation (100), graphic representation (99), measurement (99), algebra (99), geometry (99), trigonometry (91), analytic geometry (88), calculus (65), finance (94), consumer mathematics (96), probability and statistics (98), and miscellaneous (97). (The numbers indicate the "index values.")

In conclusion, Dr. Woodby recommended "that the list of 1077 topics . . . be considered as a source of topics in mathematics by teachers, curriculum workers, and writers of textbooks." (Page 221)

<sup>2</sup> National Council of Teachers of Mathematics. Fifteenth Yearbook. *The Place of Mathematics in Secondary Education*. Teachers College, Columbia University, New York, 1940.

<sup>3</sup> The Consumer Education Study; *The Role of Mathematics in Consumer Education*. Washington, D. C. 1945.

<sup>4</sup> "Essential Mathematics for Minimum Army Needs." *The MATHEMATICS TEACHER*. XXXVI, (October 1943), 243-82.

<sup>5</sup> Board of Education of the City of New York. *General Mathematics for the 9th Year*. Curriculum Bulletin, 1949-50 Series, No. 4. Brooklyn, 1950.

**The Question:** Does the study of solid geometry improve space perception abilities?

**The Study:** Ranucci, Ernest R. *The Effect of the Study of Solid Geometry on Certain Aspects of Space Perception Abilities*. Ph.D. dissertation. Teachers College, Columbia University, 1952.

For at least the past thirty years leaders in the field of the teaching of mathematics have claimed that the improvement in space perception abilities is one sound reason for the study of solid geometry. Dr. Ranucci's study was designed to test this hypothesis.

From the results of factor analysis studies in psychology the investigator found that there was considerable agreement on certain space factors present "in tasks requiring the manipulation of visual patterns." One factor "depends upon the ability to recognize objects when viewed from different positions." A second involves "the sensing of mirrored or reversed relationships." A third depends on the "body orientation of the observer." A fourth demands the ability to imagine "movement within the parts of a configuration."

To test these factors four evaluation instruments were selected from a list of eighteen. These were Thurstone's Lozenge Test A; The Revised Minnesota Paper Form Board Test, Form MA; Army General Classification Test, Form AH (Block Counting Section only); Space Relations Test, Form A, Differential Aptitudes. The experimental group from eleven high schools in northern New Jersey consisted of 225 seniors about to begin a course in

(Continued on page 626)

Wisconsin Cooperative Educational Planning Program. *General Mathematics in the High School*. Mathematics Bulletin No. 2. Curriculum Bulletin No. 17. Madison, 1950.

Florida State Department of Education. *Functional Mathematics in the Secondary Schools*. Bulletin No. 36. Tallahassee, 1950.

<sup>4</sup> Curtis, Francis S. *A Synthesis and Evaluation of Subject-Matter Topics in General Science*. Ginn and Company, Boston, 1929.

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## AIDS TO TEACHING

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*Edited by*

HENRY W. SYER  
*School of Education*  
*Boston University*  
*Boston, Massachusetts*

*and*

DONOVAN A. JOHNSON  
*College of Education*  
*University of Minnesota*  
*Minneapolis, Minnesota*

### BOOKLETS

*B. 110—Guidance Pamphlet in Mathematics*  
*B. 111—Number Stories of Long Ago*  
*B. 112—Numbers and Numerals*  
*B. 113—A Handbook on Student Teaching in Mathematics*

National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

Booklets; Described below.

*Description of B. 110:* (\$25 each, 10 or more \$10 each) This 25-page (7" x 10") booklet was originally printed in the November, 1947, issue of *THE MATHEMATICS TEACHER* as the final report of the Commission of Post-War Plans. However, it has little to do with war or post-war; it is essentially a collection of very useful, definite facts on vocational uses of mathematics. It contains the ubiquitous "check-list of 29 items" which, by being definite, has been overworked. There is now added a one-page copy of the "How High School Mathematics Can Contribute to your Career" chart prepared in Michigan.

*Description of B. 111:* (\$75 each) This reprint of David Eugene Smith's 150-page (5" x 7") book is a welcome friend to have available again. There are many illustrations and 8 pages of pictures in color. Types of numbers and types of computing from other countries and other centuries are discussed. Each chapter ends with a "Question Box" and the book with a set of "Curious Problems before the Log Fire."

*Description of B. 112:* (\$35 each) David Eugene Smith and Jekuthiel Ginsberg wrote this for the National Council in 1937. It is still an outstanding collection of pictures and discussions (52 pages, 5 1/4" x 8") on the history of mathematics, organized around the following topics: learning to count, naming the numbers, from numbers to numerals, from numerals to computation, fractions, mystery of numbers, number pleasantries, and the story of a few arithmetic words.

*Description of B. 113:* (\$15 each, 10 or more \$10 each) This booklet (34 pages, 7" x 10") is intended to help beginning student teachers and their supervisors organize their observation and teaching. It covers such topics as the following: why have student teaching, who shall take it, what issues confront teachers, what mathematics is of most worth, how to teach live mathematics, the phases of the students' experience, and how to evaluate teaching.

*Appraisal of B. 110:* The guidance pamphlet is still of great value and should be available to all secondary school students. It should not be passively introduced but actively made the part of several assignments. We are not really teaching mathematics unless we find the time to connect it with its purposes, and vocational use is one of these purposes. Sets for the whole classes, copies for the school library, and copies which students may buy if they wish, should be available in every school.

*Appraisal of B. 111 and B. 112:* These two booklets serve the same purpose, but

at quite different levels. The first is for elementary school pupils in the upper grades and for junior high schools, and the second for high school and college classes. They both provide background in the notational and computational history of arithmetic, told in a popular yet accurate style. The interest is increased by the unusually good selection of illustrations. The present reviewer cannot help but repeat his first impressions of "Number Stories" when he read it at about the age of twelve. He thought, "What an imagination this author has to invent so many ingenious ways that people *might* have done their computation." It was not until years later that he realized the methods were actual ones which had been used by the people described.

*Appraisal of B. 113:* A handbook for student teachers is an excellent idea, but this one does not quite fill the purpose. It is a summary of many excellent things which student teachers should be reminded of just before they begin that work, but the material certainly will have been covered by courses in general methods of teaching and in methods of teaching mathematics, which will usually precede student teaching. This nourishes the story that education courses say the same thing over and over. A booklet which attacks the special problems which do not arise until one faces a class would be more useful. Within its own sphere it is a useful booklet.

#### B. 114—Puzzle Craft (Kit U)

Cooperative Recreation Service, Delaware, Ohio

Booklet;  $3\frac{3}{4}'' \times 6\frac{3}{4}''$ ; 25 pages; \$25 each or in sets; \$15 in dozen lots.

*Description:* Here are 40 puzzles which can be made of wire, wood or string. Many of them have mathematical implications; for example, the devil's needle, the pyramid, the gwa, and circling dots.

*Appraisal:* This is a good collection and many items are well worth making to start a collection of puzzles. The direc-

tions are usually neither complete nor clear, but none are so difficult that a little thought will not tell how to construct them. Well worth the quarter.

#### B. 115—Bearing Load Computation

New Departure, Bristol, Conn.

Booklet;  $8\frac{1}{2}'' \times 11''$ ; 24 pages; Free in single copies.

*Description:* This is a technical booklet explaining, according to classical statics and engineering principles, how to compute loads on shaft bearing, both radial and thrust. There are many examples of formulas, geometric diagrams, empirical constants, types of variation, trigonometric functions, systems of equations, units of measure, arithmetical computations, positive and negative numbers and reading of tables of numbers.

*Appraisal:* Of course this is too technical and too detailed for any high school course in mathematics, and yet it shows on every page the outgrowth of high school mathematics. If teachers could think through just a few of the problems illustrated here and lead pupils through them pointing out the applications of the mathematics they already know, it would be more inspiration than any amount of empty reassurance that "teacher knows this is useful." Examples of this kind take time, effort and patience, but they pay off at the end.

#### B. 116—Men of Vision

Better Vision Institute, Inc., Suite 3157; 630 Fifth Ave., New York 20, N. Y.

Thirty booklets; each  $3\frac{1}{2}'' \times 5''$ , 4 pages; Free.

*Description:* Each pamphlet is on a different individual who has "opened the eyes of the world." Men who may be of interest to mathematics classes include Leonardo da Vinci, Kepler, Galileo, Leuwenhoek, Thomas Young, John Dalton, Talbot and Daguerre, Louis Pasteur, Robert Bunsen, Gregory Mendel, Roentgen, and Francis Jenkins.

*Appraisal:* Each front cover has a

title and an attractive colored diagram. Inside is an illustration and a short story which ends with an artificial reminder that these were men of vision and our eyesight is very important. By getting two sets of these and judiciously cutting and pasting them, some very attractive notebooks or bulletin board displays can be built.

### CHARTS

#### *C. 40—Optical Performance Predicted on Paper*

American Optical Company, Instrument Division, Buffalo 15, N. Y.

Poster;  $8\frac{1}{2}'' \times 11''$ ; Free.

*Description and Appraisal:* This is an illustration, on graph paper, of how mathematics is used in the design of optical systems. This is a convenient size to file in letter folders and thus have available to use year after year. Such materials can be added to as other charts, posters and clippings on the same subject are collected.

### INSTRUMENTS

#### *I. 37—Ken-Add Pocket Adding Machine*

Ken-Add Machines, P.O. Box 2, Duluth, Minn.

Calculator;  $2\frac{1}{2}'' \times 5'' \times \frac{1}{2}''$ ;  $6\frac{1}{2}$  ounces; \$6.95.

*Description:* There are four columns which give this device a capacity of 9999. The carry-principle is incorporated and is mechanically very smooth, effective, and seemingly infallible. There is a stylus which is used to dial the figures and also to write on the magic slate that is inside the cover. The outside is a stippled metal finish.

*Appraisal:* This is one of the most attractive of the simple addition devices in this price class. It is not a toy, nor primarily a device for schools, but a practical adding machine. Nevertheless, it is as fascinating as a toy, and very useful in many arithmetic situations in school. It will find value in teaching place value, carrying from one column to another,

checking addition problems, checking subtraction problems, illustrating multiplication as repeated addition, and division as repeated subtraction, and many other learnings as well. With ordinary precaution it seems very durable and beyond most of the abuses of carelessness. The magic slate can be used to record results of operations which are not permanent enough to transfer to paper. A few of these calculators should be in the teaching collection of each school to enrich the teaching of computation. In fact, with a few different devices of this sort and supplementary reading matter, a very good unit on calculating devices could be organized at the secondary school level.

#### *I. 38—Judy Clock*

The Judy Company, 310 North Second Street, Minneapolis 1, Minn.

Clock face;  $12\frac{1}{2}'' \times 13''$ ; \$3.00 (school discount).

*Description:* This clock face is stenciled in very attractive colors and is supported on two  $5''$  legs which are easily detachable for storage. The hands are  $4\frac{1}{2}''$  and  $6\frac{1}{2}''$ . This instrument differs from others in having a small window set in the front where a set of gears can be seen turning which keep the true relationship between the two hands when just the minute hand is rotated. Pictures of a sundial and an hour glass decorate the surface.

*Appraisal:* Only by extended trial could it be determined whether the gear mechanism would stand up under continual usage and thus justify the additional complication. The gears do certainly add to the ease of setting the clock and show the pupils that a similar relationship holds in real clocks. This is an attractive addition to any teaching collection and very practical.

### MODELS

#### *M. 29—Mek-N-Ettes, Jr.*

The Judy Company, 310 North Second Street, Minneapolis 1, Minn.

Construction set; Box,  $7\frac{1}{2}'' \times 10''$ ; \$1.75.

**Description:** This is a smaller version of the Deluxe Mek-N-Ettes reviewed previously (M. 23, May 1951) with none of the complex parts such as pistons, cams, etc., which the larger set has. The base of this set is  $7\frac{1}{2}'' \times 10''$  and has 59 holes one inch apart on a diagonal grid. Shafts, 1" and 3" gears,  $\frac{1}{2}$ ",  $\frac{3}{4}$ ",  $1\frac{1}{4}$ " fiber washers, and small rubber washers are provided. Many different gear trains can be demonstrated.

**Appraisal:** The science department can not afford to be without these devices. Once the mathematics teachers have tried them, themselves or with their pupils, they will be convinced that there is both fun and very good mathematics involved. In setting up and figuring out the gear trains one will meet the counting of gear teeth, the ratio of gear teeth, the ratio of number of revolutions, the distances between centers, the need to adjust thicknesses so that teeth will mesh, and other mathematical concepts. By all means have one of the large Deluxe sets for each school and enough Mek-N-Ettes, Jr., so that groups of students can work out problems on them.

#### M.30—Curve Unit

National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

Construction Kit;  $4\frac{1}{2}'' \times 7''$ ; \$ .75 each, 3 for \$1.50.

**Description:** This envelope contains another unit developed by the National Council in cooperation with Science Service and was issued as #139 of the Things of Science series. The others previously reviewed were the Geometric Models Kit (M. 7, Nov. 1949), the Straight Line Unit (M. 13, Nov. 1950), and the Computation Unit (Nov. 1951). The supply of the first three is completely exhausted. The present unit contains a sheet of waxed paper ( $8'' \times 12''$ ), 2 skeins of red and blue embroidery thread (8 yards each), an embroidery needle (a real puzzle to find where it is hidden!), 6 pieces of cardboard

( $4\frac{1}{4}'' \times 6\frac{1}{4}''$ ) to serve as bases for curve stitching and a three-dimensional cylinder-hyperboloid-cone model, and a booklet of directions ( $5'' \times 6''$ , 10 pages).

The curves which are constructed by curve stitching, paper folding, string and paper construction, and cutting a string model of a cone are all conic sections (ellipse, parabola, and hyperbola). There are 22 "experiments" outlined to obtain these curves in various ways.

**Appraisal:** This is a clever and colorful kit. The method of obtaining the string model in three dimensions is especially ingenious. The models obtained are small, but sturdy, and should certainly serve merely as an inspiration to make larger, more colorful ones with even stronger materials. It is unfortunate that a short bibliography could not have been included in the booklet to send people to other sources; they certainly will want to find them. The price is high, but fair. After the clear help of this kit, teachers and pupils can go on to make more models with odds and ends of materials at a much cheaper rate.

### SOURCES OF MATERIAL FOR LABORATORY WORK

#### SL. 26—*Investigations into Geometry*

Educational Enterprises, Wayne, N. J.

Notebook and devices;  $8\frac{1}{2}'' \times 11''$ ; \$2.00 each, 4 or more copies, \$1.80 each. (May be purchased separately.)

**Description:** There are 57 pages with 44 "investigations" printed on them to serve as the nucleus of a laboratory course in plane geometry. They cover the topics of angles, triangles, parallel lines, angles of polygons, parallelograms, circles, inequalities, angles and circles, similar triangles, products and proportions, right triangles, trigonometric functions, and area of polygons. The investigations are in the form of work sheets with spaces and tables to be filled in. The figures to fill in the tables, and thus develop inductive conclusions,

result from the measurement of five heavy cardboard models which are included in the notebook material: an angle device, two triangles, a parallel line device, and a circle device.

The angle, parallel line and circle devices are mounted on heavy cardboard which is punched to fit the notebook. The two triangles are in a punched envelope. The fasteners at the corners of the triangles may be taken apart so that any polygon up to a hexagon can be constructed. The sides and angles in all the figures can be varied continuously to study the functional relationships between them. In fact the devices are similar to the Schacht models (M. 2, Dec. 1948; M. 6, May 1949) and the work sheets to the book of *Plane Geometry Experiments* published by Van Nostrand (Sl. 19, May 1950).

*Appraisal:* This is excellent material and should be given a fair try by every teacher. Every fact in the plane geometry course need not be developed inductively, but the method should have its place to show the plausibility of the conclusions

and the functional relationships between the geometric parts of the figures. All geometry is not metric, but the metric can be taught better than it is. The form of the "investigations" and the choice of cardboard devices are conducive to good teaching; they actually should lead the pupil to do some thinking for himself.

The most obvious comparison which will constantly be made is between these cardboard devices and the Schacht devices. The conclusion is simple: the Schacht devices are more permanent and more expensive, these are more fragile and less expensive. There seems no way to determine which will be more economical in the long run without testing them both to destruction. The Schacht devices are more accurate, but these appeal for their simplicity. Why not get one set of Schacht devices and a lot of these for a whole school system and systematically circulate them to all geometry teachers, finally taking a vote? There is so much of this worth-while material now being manufactured that every teacher should give it an open-minded trial.

### Research

(Continued from page 621)

solid geometry, with two years of algebra and one year of plane geometry as a background. The four tests were given to these students and to a control group, in the same schools, having the same mathematical foundation. The tests were administered early in the term and five months later.

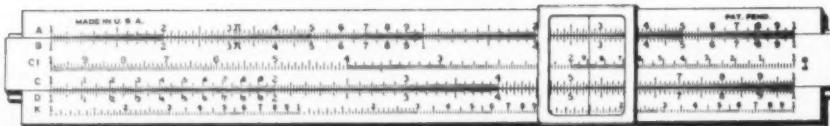
When the "t" test gains of the two groups over the five months period were compared, the results indicated that two different populations were involved. Hence, matched groups were secured on the basis of school attended, sex, I.Q. and whether or not one term of mechanical drawing or none at all had been studied. Of course, as indicated

above, both groups had the same mathematical preparation as far as names of courses were concerned. The findings were startling: no statistically significant differences were found as a result of using the "Sign Test" and "Paired Replicates Test." This conclusion was not changed even when the higher initial scores of the experimental group were taken into account.

Among Dr. Ranucci's conclusions were these (page 56): "The claim of many that the study of solid geometry will improve space perception abilities, . . . , has little statistical backing." "If the improvement of space perception abilities, as evidenced by performance on the battery of tests administered, is deemed to be one of the important outcomes of teaching solid geometry, this aim is not being met."

**Index photographed at the  
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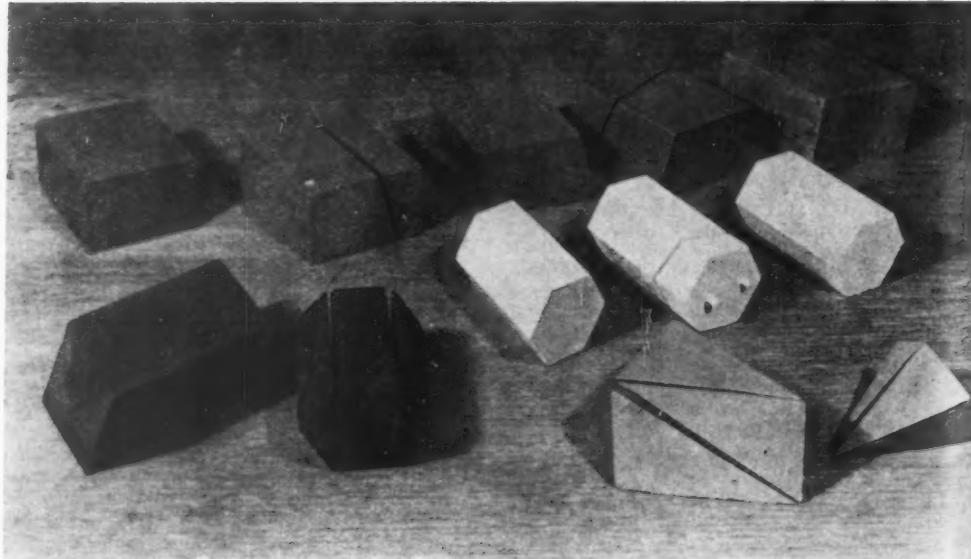
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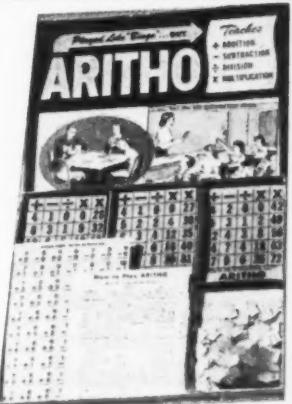
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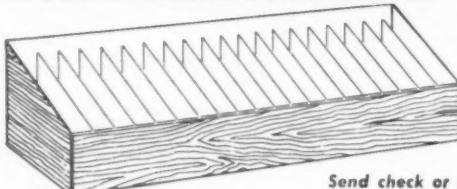
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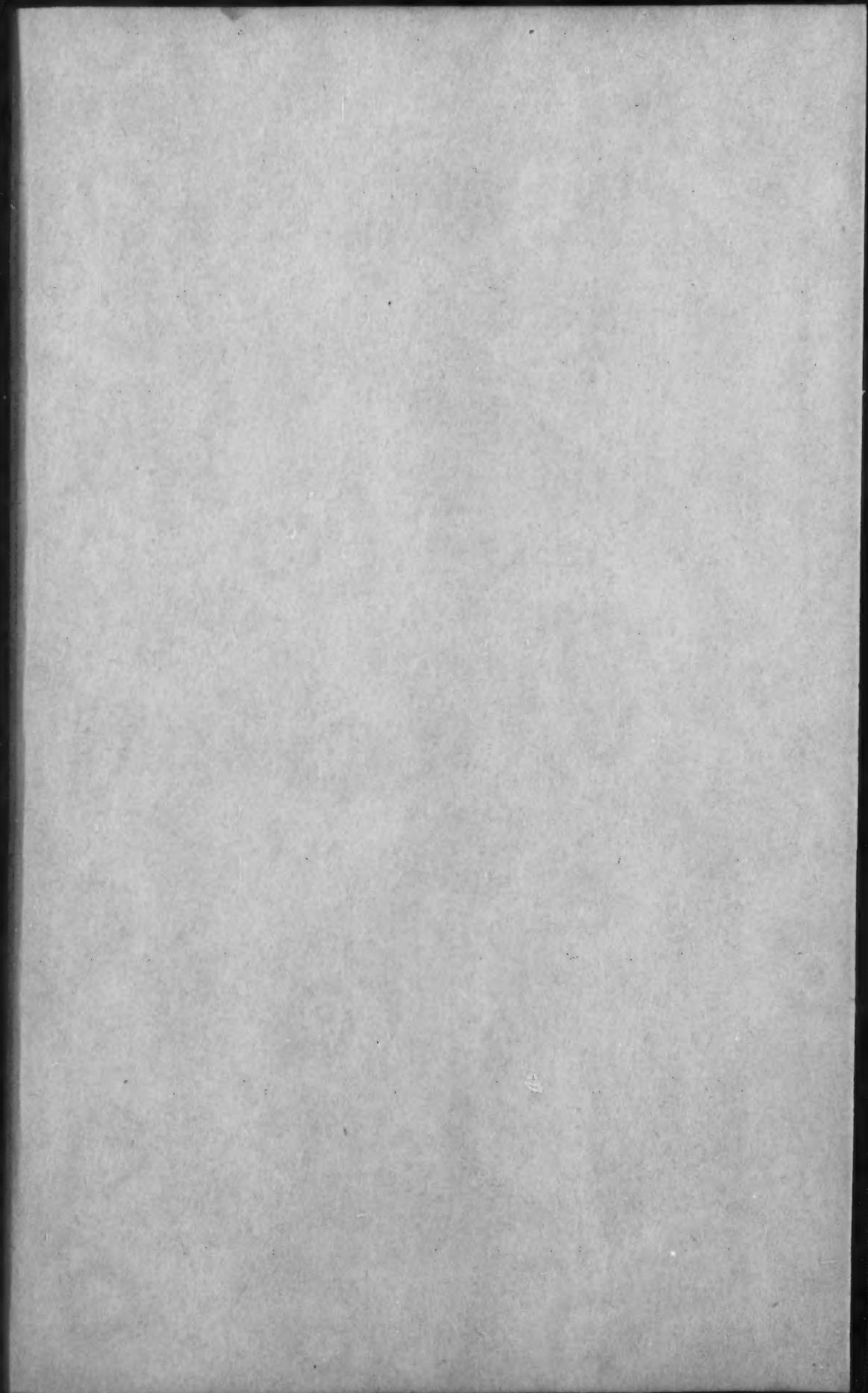
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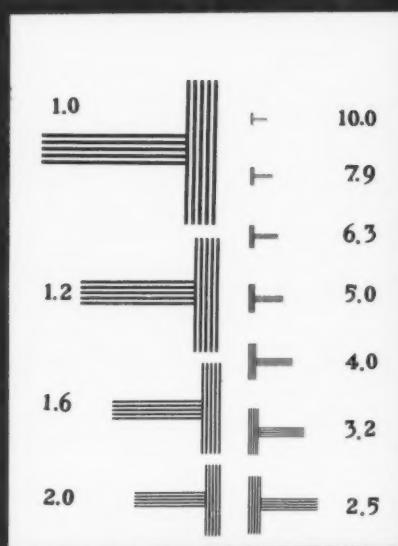
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# RESOLUTION CHART



100 MILLIMETERS

**INSTRUCTIONS** Resolution is expressed in terms of the lines per millimeter recorded by a particular film under specified conditions. Numerals in chart indicate the number of lines per millimeter in adjacent "T-shaped" groupings.

In microfilming, it is necessary to determine the reduction ratio and multiply the number of lines in the chart by this value to find the number of lines recorded by the film. As an aid in determining the reduction ratio, the line above is 100 millimeters in length. Measuring this line in the film image and dividing the length into 100 gives the reduction ratio. Example: the line is 20 mm. long in the film image, and  $100/20 = 5$ .

Examine "T-shaped" line groupings in the film with microscope, and note the number adjacent to finest lines recorded sharply and distinctly. Multiply this number by the reduction factor to obtain resolving power in lines per millimeter. Example: 7.9 group of lines is clearly recorded while lines in the 10.0 group are not distinctly separated. Reduction ratio is 5, and  $7.9 \times 5 = 39.5$  lines per millimeter recorded satisfactorily.  $10.0 \times 5 = 50$  lines per millimeter which are not recorded satisfactorily. Under the particular conditions, maximum resolution is between 39.5 and 50 lines per millimeter.

Resolution, as measured on the film, is a test of the entire photographic system, including lens, exposure, processing, and other factors. These rarely utilize maximum resolution of the film. Vibrations during exposure, lack of critical focus, and exposures yielding very dense negatives are to be avoided.

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